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Perturbation bounds for $GI/M/s$ queue : The Strong Stability Method

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Résumé This paper investigates when the $M/M/s$ model can be used to predict an estimate for the proximity of the performance measures of queues with arrival processes that are slightly different from the Poisson process assumed in the model. The arrival processes considered here are perturbed Poisson processes. The perturbations are deviations from the exponential distribution of the inter-arrival times or from the assumption of independence between successive inter-arrival times.
In this work, we apply the strong stability method to obtain an estimate for the proximity of the performance measures in the $GI/M/s$ queueing system to the same performance measures in the $M/M/s$ system under the assumption that the distributions of the arrival time are close and the service flows coincide. In addition to the proof of the stability fact for the perturbed $M/M/s$ queueing system, we obtain the inequalities of the stability. These results give with precision the error, on the queue size stationary distribution, due to the approximation.

Keywords : Markov chains, Strong stability, queueing System.

8.1 Notations and Preliminaries

In this section we introduce necessary notations. For the basic theorems of the strong stability method are given in [1]. The main tool for our analysis is the weighted supermum norm, also called υ-norm, denoted by $\|.\|_\upsilon$, where υ is some vector with elements $\upsilon(k) \geq 1$ for all $k \in \mathbb{Z}_+$, and for any vector f with infinite dimension

$$\|f\|_\upsilon = \sup_{k\geq 0} \frac{|f(k)|}{\upsilon(k)}. \tag{8.1}$$

Let μ be a probability measure on $\mathbb{Z}_+$, then the υ-norm of μ is defined as

$$\|\mu\|_\upsilon = \sum_{j\geq 0} \upsilon(j)|\mu_j|. \tag{8.2}$$

The υ-norm is extended to stochastic kernels on $\mathbb{Z}_+$ in the following way : let P the matrix with infinite dimension then

$$\|P\|_\upsilon = \sup_{k\geq 0} \frac{\|P(k,.)\|_\upsilon}{\upsilon(k)} = \sup_{k\geq 0} \frac{1}{\upsilon(k)} \sum_{j\geq 0} \upsilon(j)|P_{kj}|. \tag{8.3}$$

Note that υ-norm convergence to 0 implies elementwise convergence to 0.

We associate to each transition kernel P the linear mappings :

$$(\mu P)_k = \sum_{i\geq 0} \mu_i P_{ik}. \tag{8.4}$$

$$(Pf)(k) = \sum_{i\geq 0} f(i) P_{ki}. \tag{8.5}$$

The strong stability method [2, 1] considers the problem of the perturbation of general state space Markov chains using operator's theory and with respect to a general class of norms. The basic idea behind the concept of stability is that, for a strongly stable Markov chain, a small perturbation in the transition kernel can lead to only a small deviation of the stationary distribution.

Définition 8.1 *A Markov chain X with transition kernel P and stationary distribution π is said to be strongly stable with respect to the norm $\|.\|_\upsilon$ if $\|P\|_\upsilon < \infty$ and every stochastic kernel Q in some neighborhood $\{Q \ : \ \|Q-P\|_\upsilon \leq \varepsilon\}$ admits a unique stationary distribution ν and*

$$\|\nu - \pi\|_\upsilon \to 0 \quad as \quad \|Q-P\|_\upsilon \to 0. \tag{8.6}$$

In fact, as shown in [2], X is strongly stable if and only if, there exists a positive constant $c = c(P)$ such that

$$\|\nu - \pi\|_\upsilon \leq c\|Q-P\|_\upsilon. \tag{8.7}$$

In the sequel we use the following results.

Théorème 8.1 ([2]) *The Markov chain X with the transition kernel P and stationary distribution π is strongly stable with respect to the norm $\|.\|_\upsilon$, if and only if there exists a probability measure $\alpha = (\alpha_j)$ and a vector $h = (h_i)$ on $\mathbb{Z}_+$ such that $\pi h > 0$, $\alpha \mathbb{1} = 1$, αh is a positive scalar, and*

a. The matrix $T = P - h\alpha$ is non-negative, where $h\alpha = (a_{ij})_{ij}$ such that $a_{ij} = h_i \alpha_j$ for $i, j \in \mathbb{Z}_+$.

b. There exists $\rho < 1$ such that $T\upsilon(k) \leq \rho\upsilon(k)$ for $k \in \mathbb{Z}_+$.

c. $\|P\|_\upsilon < \infty$.

Here $\mathbb{1}$ is the vector having all the components equal to 1.

Théorème 8.2 ([1]) *Let X be a strongly υ-stable Markov chain that satisfies the conditions of Theorem 8.1. If ν is the probability invariant measure of a stochastic kernel Q, then for $\|\Delta\|_\upsilon < (1-\rho)/c$, we have the estimate*

$$\|\nu - \pi\|_\upsilon \leq c\|\Delta\|_\upsilon \|\pi\|_\upsilon (1 - \rho - c\|\Delta\|_\upsilon)^{-1}, \tag{8.8}$$

where $\Delta = Q - P$, $c = 1 + \|\not\Pi\|_\upsilon \|\pi\|_\upsilon$ and $\|\pi\|_\upsilon \leq (\alpha\upsilon)(1-\rho)^{-1}(\pi h)$.

8.2 Analysis of the Model

8.2.1 Model Description

We consider a $GI/M/s$ queueing system (s servers) with infinite capacity. Customers arrive at time points $t_0 = 0$, t_1, t_2, $\ldots$ where the interarrival times $Z_n = t_{n+1} - t_n$, $n = 1, 2, 3, \ldots$, are independent identically distributed random variables (i.i.d r.v.'s) having a non lattice c.d.f $H(\cdot)$ with mean γ^{-1}. The sevice times S_1, S_2, $\ldots$ are i.d.d r.v.'s having a common exponentiel d.f with a finite mean μ^{-1}. Let $\gamma/s\mu$ be the traffic intensity, assumed to be strictly less than one. Let $\widetilde{Q}(t)$ be the number of customers in the system at time t and define $\widetilde{Q}(t_n - 0) = \widetilde{Q}_n$, $n = 1, 2, \ldots$. Thus $\widetilde{Q}_n$ is the number in the system just before the nth arrival. Now consider the relationship between $\widetilde{Q}_n$ and $\widetilde{Q}_{n+1}$. We have

$$\widetilde{Q}_{n+1} = \begin{cases} \widetilde{Q}_n + 1 - \widetilde{X}_{n+1} & \text{if } \widetilde{Q}_n + 1 - \widetilde{X}_{n+1} > 0, \\ 0 & \text{if } \widetilde{Q}_n + 1 - \widetilde{X}_{n+1} \leq 0, \end{cases} \tag{8.9}$$

where $\widetilde{X}_{n+1}$ is the total number of potential customers who can be served by s servers during an interarrival period Z_n. Due to the exponentiel service time, the process $\{\widetilde{Q}_n, n = 0, 1, 2, \ldots\}$ is an homogeneous Markov chain. From (8.9), it is found that the evolution of the homogeneous Markov chain $(\widetilde{Q}_n)_{n\geq 1}$ is governed by the transition probability matrice $\widetilde{P} = (\widetilde{P}(i,j))_{i,j\geq 0}$ described by

$\widetilde{P}(i,j) = 0 \quad (i + 1 - j < 0).$

$\widetilde{P}(i,j) = \displaystyle\int_0^\infty \frac{(s\mu t)^{i+1-j}}{(i+1-j)!} e^{-s\mu t} dH(t) \quad (i \geq s-1,\ j \geq s,\ i+1-j \geq 0).$

$\widetilde{P}(i,j) = \displaystyle\int_0^\infty \binom{i+1}{i+1-j} e^{-j\mu t}(1 - e^{-\mu t})^{i+1-j} dH(t) \quad (i \leq s-1,\ i+1-j \geq 0)$ $\widetilde{P}(i,j) = \displaystyle\int_0^\infty \int_{\tau=0}^t \binom{s}{s-j}$

$(i \geq s,\ j < s,\ i+1-j \geq 0).$

Consider also an system $M/M/s$, which has the same distribution of service times, where the interarrival times are independent identically distributed random variables and vary according to an exponential distribution $E_\lambda(\cdot)$ with a finite mean λ^{-1}. Further, the embedded Markov chain $(Q_n)_{n\geq 1}$, representing the number of customers in the $M/M/s$

queueing system. Denote by $P = (P(i,j))_{i,j\geq 0}$ the transition operators of the Markov chains $(Q_n)_{n\geq 1}$.

8.2.2 υ-Strong Stability Conditions

the main work in strong stability method is finding β such that $\|T\|_\upsilon < 1$, where T is a stochastic kernel. For that, we choose the function $\upsilon(k) = \beta^k$, $\beta > 1$, $h_i = \mathbf{I}_{i=0}$ and $\alpha_j = P_{0j}$ (see Theorem 8.1).

Théorème 8.3 *Suppose that in the $M/M/s$ queueing system the following geometric ergodicity condition, $\lambda/s\mu < 1$, holds. Then for all $\beta \in \mathbb{R}$ such that, $1 < \beta < \beta_0$, the embedded Markov chain $(Q_n)_{n\geq 1}$ is υ-strongly stable for the test function $\upsilon(k) = \beta^k$.*

Proof. We have $\pi h = \pi_0 > 0$, $\alpha \mathbb{1} = 1$ and $\alpha h = \alpha_0 = P_{00} > 0$.

$$T_{ij} = P_{ij} - h_i \alpha_j = \begin{cases} 0, & \text{if } i = 0, \\ P_{ij}, & \text{if } i \geq 1. \end{cases} \tag{8.10}$$

Hence, the kernel T is nonnegative.
According to Equation (8.5), we have :

$$T\upsilon(i) = \sum_{j\geq 0} \beta^j T_{ij}. \tag{8.11}$$

(a) If $i = 0$, then

$$T\upsilon(0) = \sum_{j\geq 0} \beta^j T_{0j} = 0. \tag{8.12}$$

(b) If $1 \leq i \leq s-2$, then

$$\begin{aligned} T\upsilon(i) = \sum_{j=0}^{i+1} \beta^j P_{ij} &= \beta^{i+1} \int_0^\infty \sum_{j=0}^{i+1} \binom{i+1}{j} \left(\frac{1-e^{-\mu t}}{\beta}\right)^{i+1-j} (e^{-\mu t})^j dE_\lambda(t) \\ &\leq \beta^i \int_0^\infty \frac{1}{\beta} \left(1 + (\beta-1)e^{-\mu t}\right)^2 dE_\lambda(t) \end{aligned}$$

$$\text{We pose,} \quad \rho_1 = \int_0^\infty \frac{1}{\beta} \left(1 + (\beta-1)e^{-\mu t}\right)^2 dE_\lambda(t) = \int_0^\infty f(t) dE_\lambda(t)$$

(c) If $i = s-1$, then

$$\begin{aligned}
Tv(s-1) = \sum_{j=0}^{s} \beta^j P_{(s-1)j} \quad &= \quad \sum_{j=0}^{s-1} \beta^j P_{(s-1)j} + \beta^s P_{(s-1)s} \\
&= \sum_{j=0}^{s-1} \beta^j \int_0^\infty \binom{s}{s-j} (1-e^{-\mu t})^{s-j} (e^{-\mu t})^j dH(t) + \beta^s \int_0^\infty e^{-s\mu t} dE_\lambda(t) \\
&= \beta^s \int_0^\infty \left(\frac{1-e^{-\mu t}}{\beta} + e^{-\mu t} \right)^s dE_\lambda(t) \\
&\leq \beta^{s-1} \int_0^\infty \frac{1}{\beta} \left(1 + (\beta-1)e^{-\mu t}\right)^2 dE_\lambda(t) \quad = \quad \beta^{s-1}\rho_1
\end{aligned}$$

(d) If $i \geq s$, then

$$Tv(i) = \sum_{j=0}^{i+1} \beta^j P_{ij} \quad = \quad \sum_{j=0}^{s-1} \beta^j P_{ij} + \sum_{j=s}^{i+1} \beta^j P_{ij}$$

$$\begin{aligned}
\sum_{j=0}^{s-1} \beta^j P_{ij} &= \sum_{j=0}^{s-1} \beta^j \int_0^\infty \int_{\tau=0}^{t} \binom{s}{s-j} e^{-j\mu(t-\tau)} (1-e^{-\mu(t-\tau)})^{s-j} \frac{(s\mu\tau)^{i-s}}{(i-s)!} e^{-s\mu\tau} s\mu d\tau dE_\lambda(t) \\
&\leq \beta^{s-1} \int_0^\infty \Bigg[e^{-s\mu t} \sum_{n=i+1-s}^{\infty} \frac{(s\mu t)^n}{n!} + (\beta-1) e^{-s\mu t} \big(\frac{s}{s-1}\big)^{i+1-s} \sum_{n=i+1-s}^{\infty} \frac{((s-1)\mu t)^n}{n!} \\
&\quad -\beta e^{-s\mu t} \frac{(s\mu t)^{i+1-s}}{(i+1-s)!} \Bigg] dE_\lambda(t)
\end{aligned}$$

$$\text{And,} \quad \sum_{j=s}^{i+1} \beta^j P_{ij} = \beta^{i+1} \int_0^\infty e^{-s\mu t} \sum_{n=0}^{i+1-s} \frac{(s\mu t/\beta)^n}{n!} dE_\lambda(t)$$

Therefore,

$$\begin{aligned}
Tv(i) &= \sum_{j=0}^{s-1} \beta^j P_{ij} + \sum_{j=s}^{i+1} \beta^j P_{ij} \\
&\leq \beta^i \int_0^\infty \left(\frac{1}{\beta} - \frac{1}{\beta} e^{-s\mu t} + \big(\frac{\beta-1}{\beta}\big) \big(\frac{s}{s-1}\big) (e^{-\mu t} - e^{-s\mu t}) + \beta e^{-s\mu t} \right) dE_\lambda(t)
\end{aligned}$$

We pose,

$$\rho_2 = \int_0^\infty \left(\frac{1}{\beta} - \frac{1}{\beta} e^{-s\mu t} + \big(\frac{\beta-1}{\beta}\big) \big(\frac{s}{s-1}\big) (e^{-\mu t} - e^{-s\mu t}) + \beta e^{-s\mu t} \right) dE_\lambda(t) = \int_0^\infty g(t) dE_\lambda(t)$$

We have, with assumption that $s \geq 2$, $\dfrac{s}{s-1} = 1 + \dfrac{1}{s-1} \leq 2$.

$$\text{Then, } g(t) = \frac{1}{\beta} - \frac{1}{\beta}e^{-s\mu t} + \big(\frac{\beta-1}{\beta}\big)\big(\frac{s}{s-1}\big)(e^{-\mu t} - e^{-s\mu t}) + \beta e^{-s\mu t} \tag{8.13}$$

$$\leq \frac{1}{\beta}\left(1 + (\beta-1)e^{-\mu t}\right)^2 \quad = \quad f(t) \tag{8.14}$$

this shows that $\rho_2 = \displaystyle\int_0^\infty g(t)dE_\lambda(t) \leq \int_0^\infty f(t)dE_\lambda(t) = \rho_1$.
It suffices to take, $\rho = \max(\rho_1, \rho_2) = \displaystyle\int_0^\infty \frac{1}{\beta}\left(1 + (\beta-1)e^{-\mu t}\right)^2 dE_\lambda(t)$ which is smaller then 1 for all $\beta > 1$. Now, we have $E_\lambda(t) = 1 - e^{-\lambda t}$, then

$$\rho = \frac{1}{\beta} + \frac{2\lambda(\beta-1)}{\beta(\lambda+\mu)} + \frac{\lambda(\beta-1)^2}{\beta(2\mu+\lambda)} \tag{8.15}$$

And, with assumption that $\beta > 1$, We have $\rho < 1 \;\Rightarrow\; \beta < \dfrac{2\mu^2}{\lambda(\lambda+\mu)}$.
We pose, $\beta_0 = \dfrac{2\mu^2}{\lambda(\lambda+\mu)}$. Then, for all β such that $1 < \beta < \beta_0$, we have $\rho < 1$.
Now, we verify that $\|P\|_\upsilon < \infty$. We have

$$T = P - h\alpha \;\Rightarrow\; P = T + h\alpha \;\Rightarrow\; \|P\|_\upsilon \leq \|T\|_\upsilon + \|h\|_\upsilon\|\alpha\|_\upsilon, \tag{8.16}$$

or, according to equation (8.3),

$$\|T\|_\upsilon = \sup_{i\geq 0}\frac{1}{\upsilon(i)}\sum_{j\geq 0}\upsilon(j)|T_{ij}| \leq \sup_{i\geq 0}\frac{1}{\upsilon(i)}\rho\upsilon(i) \leq \rho < 1. \tag{8.17}$$

According to Equations (8.1) and (8.2), we have : $\|h\|_\upsilon = \sup\limits_{i\geq 0}\dfrac{1}{\upsilon(i)}|h_i| = 1$,

And, $\|\alpha\|_\upsilon = \displaystyle\sum_{j\geq 0}\upsilon(j)|\alpha_j| \;=\; \sum_{j\geq 0}\beta^j P_{0j} \;=\; P_{00} + \beta P_{01} \;<\; \beta(P_{00}+P_{01}) \;\leq\; \beta \;<\; \infty.$

Then, $\|P\|_\upsilon < \infty$.
The Markov chain $(Q_n)_{n\geq 1}$ being strongly stable then, the $\|\pi - \widetilde{\pi}\|_\upsilon$ can be bounded in terms of $\|P - \widetilde{P}\|_\upsilon$.

8.2.3 Bound on Perturbation

To be able to estimate numerically the margin between the stationary distributions of the Markov chains $(\widetilde{Q}_n)_{n\geq 1}$ and $(Q_n)_{n\geq 1}$ we estimate the norm of the deviation of the transition kernel.

Lemme 8.1 *Let $\widetilde{P}$ (respectively P) be the transition kernel of the Markov chain $(\widetilde{Q}_n)_{n\geq 1}$ (respectively of the Markov chain $(Q_n)_{n\geq 1}$). Then, for all β such that $1<\beta<\beta_0$, we have :*

$$\|P-\widetilde{P}\|_v \leq \int_0^{\infty}\Big(1+\big(\beta-1\big)e^{-\mu t}\Big)|H-E_\lambda|(dt) \tag{8.18}$$

Proof. From Equation (8.3), we have

$$\|P-\widetilde{P}\|_v = \sup_{i\geq 0}\frac{\|P(i,.)-\widetilde{P}(i,.)\|_v}{v(i)} = \sup_{i\geq 0}\frac{1}{v(i)}\sum_{j=0}^{i+1} v(j)|P_{ij}-\widetilde{P}_{ij}|. \tag{8.19}$$

(a) For $i\leq s-2$, we have,

$$\begin{aligned}\sum_{j=0}^{i+1} v(j)|P_{ij}-\widetilde{P}_{ij}| &= \sum_{j=0}^{i+1}\beta^j|P_{ij}-\widetilde{P}_{ij}| \\ &\leq \beta^{i+1}\sum_{j=0}^{i+1}\int_0^{\infty}\binom{i+1}{i+1-j}\left(\frac{1-e^{-\mu t}}{\beta}\right)^{i+1-j}(e^{-\mu t})^j|H-E_\lambda|(dt) \\ &\leq \beta^i\int_0^{\infty}\left(1+(\beta-1)e^{-\mu t}\right)|H-E_\lambda|(dt)\end{aligned}$$

$$\text{Then,}\quad \|P-\widetilde{P}\|_v \leq \int_0^{\infty}\left(1+(\beta-1)e^{-\mu t}\right)|H-E_\lambda|(dt). \tag{8.20}$$

$$\text{We pose,}\quad \Delta_1 = \int_0^{\infty}\left(1+(\beta-1)e^{-\mu t}\right)|H-E_\lambda|(dt) = \int_0^{\infty}\zeta(t)|H-E_\lambda|(dt)$$

(b) For $i=s-1$, we have,

$$\begin{aligned}&\sum_{j=0}^{s} v(j)|P_{(s-1)j}-\widetilde{P}_{(s-1)j}| \\ &= \sum_{j=0}^{s-1}\beta^j|P_{(s-1)j}-\widetilde{P}_{(s-1)j}| + \beta^s|P_{(s-1)s}-\widetilde{P}_{(s-1)s}| \\ &\leq \sum_{j=0}^{s-1}\beta^j\int_0^{\infty}\binom{s}{s-j}(1-e^{-\mu t})^{s-j}(e^{-\mu t})^j|H-E_\lambda|(dt) + \beta^s\int_0^{\infty}e^{-s\mu t}|H-E_\lambda|(dt) \\ &= \beta^s\int_0^{\infty}\left(\frac{1-e^{-\mu t}}{\beta}+e^{-\mu t}\right)^s|H-E_\lambda|(dt) \quad\leq\quad \beta^{s-1}\int_0^{\infty}\left(1+(\beta-1)e^{-\mu t}\right)|H-E_\lambda|(dt)\end{aligned}$$

Then,

$$\|P-\widetilde{P}\|_v \leq \int_0^{\infty}\left(1+(\beta-1)e^{-\mu t}\right)|H-E_\lambda|(dt) = \Delta_1. \tag{8.21}$$

(c) For $i\geq s$, we have,

$$\sum_{j=0}^{i+1} \upsilon(j)|P_{ij}-\widetilde{P}_{ij}| = \sum_{j=0}^{i+1} \beta^j |P_{ij}-\widetilde{P}_{ij}| \quad = \quad \sum_{j=0}^{s-1} \beta^j |P_{ij}-\widetilde{P}_{ij}| + \sum_{j=s}^{i+1} \beta^j |P_{ij}-\widetilde{P}_{ij}|$$

we have,

$$\begin{aligned}
&\sum_{j=0}^{s-1} \beta^j |P_{ij}-\widetilde{P}_{ij}| \\
&\leq \beta^s \int_0^\infty \int_0^t \left[\sum_{j=0}^{s-1} \binom{s}{j} \left(\frac{1-e^{-\mu(t-\tau)}}{\beta}\right)^{s-j} \left(e^{-\mu(t-\tau)}\right)^j\right] \frac{(s\mu\tau)^{i-s}}{(i-s)!} e^{-s\mu\tau} s\mu d\tau |H-E_\lambda|(dt) \\
&\leq \beta^{s-1} \int_0^\infty \Bigg[e^{-s\mu t} \sum_{n=i+1-s}^{\infty} \frac{(s\mu t)^n}{n!} + (\beta-1)e^{-s\mu t}\big(\frac{s}{s-1}\big)^{i+1-s} \sum_{n=i+1-s}^{\infty} \frac{((s-1)\mu t)^n}{n!} \\
&\quad -\beta e^{-s\mu t} \frac{(s\mu t)^{i+1-s}}{(i+1-s)!}\Bigg] |H-E_\lambda|(dt)
\end{aligned}$$

And,

$$\begin{aligned}
\sum_{j=s}^{i+1} \beta^j |P_{ij}-\widetilde{P}_{ij}| &\leq \beta^{i+1} \int_0^\infty e^{-s\mu t} \sum_{j=s}^{i+1} \frac{(s\mu t/\beta)^{i+1-j}}{(i+1-j)!} |H-E_\lambda|(dt) \\
&= \beta^{i+1} \int_0^\infty e^{-s\mu t} \sum_{n=0}^{i+1-s} \frac{(s\mu t/\beta)^n}{n!} |H-E_\lambda|(dt)
\end{aligned}$$

therefore,

$$\sum_{j=0}^{i+1} \upsilon(j)|P_{ij}-\widetilde{P}_{ij}| \leq \beta^i \int_0^\infty \left(\frac{1}{\beta} - \frac{1}{\beta} e^{-s\mu t} + \big(\frac{\beta-1}{\beta}\big)\big(\frac{s}{s-1}\big)(e^{-\mu t} - e^{-s\mu t}) + \beta e^{-s\mu t}\right) |H-E_\lambda|(dt)$$

Then,

$$\|P-\widetilde{P}\|_\upsilon \leq \int_0^\infty \left(\frac{1}{\beta} - \frac{1}{\beta} e^{-s\mu t} + \big(\frac{\beta-1}{\beta}\big)\big(\frac{s}{s-1}\big)(e^{-\mu t} - e^{-s\mu t}) + \beta e^{-s\mu t}\right) |H-E_\lambda|(dt) \quad (8.22)$$

We pose,

$$\varDelta_2 = \int_0^\infty \left(\frac{1}{\beta} - \frac{1}{\beta} e^{-s\mu t} + \big(\frac{\beta-1}{\beta}\big)\big(\frac{s}{s-1}\big)(e^{-\mu t} - e^{-s\mu t}) + \beta e^{-s\mu t}\right) dE_\lambda(t) = \int_0^\infty g(t)|H-E_\lambda|(dt)$$

According to Equations (8.13), we have :

$$g(t) \leq \frac{1}{\beta}\left(1+(\beta-1)e^{-\mu t}\right)^2 \quad \leq \quad \left(1+(\beta-1)e^{-\mu t}\right) \quad = \quad \zeta(t)$$

this shows that $\varDelta_2 = \int_0^\infty g(t)|H-E_\lambda|(dt) \leq \int_0^\infty \zeta(t)|H-E_\lambda|(dt) = \varDelta_1$.

Finally, it suffices to take, $\|P-\widetilde{P}\|_\upsilon \leq \varDelta = \max(\varDelta_1, \varDelta_2) = \int_0^\infty \left(1+(\beta-1)e^{-\mu t}\right)|H-E_\lambda|(dt)$.

Lemme 8.2 *Let π be the stationary distribution of the embedded Markov chain $(Q_n)_{n\geq 1}$. Then, for all $1 < \beta < \beta_0$, we have :*

$$||\pi||_\upsilon = \pi_0 \left(\sum_{k=0}^{s-1} \frac{(s\varrho\beta)^k}{k!} + \frac{(\lambda\beta/\mu)^s}{s!(1-\varrho\beta)} \right) = c_0 \tag{8.23}$$

where,

$$\varrho = \frac{\lambda}{s\mu} < 1$$

and,

$$\pi_0 = \left[\sum_{k=0}^{s-1} \frac{(s\varrho)^k}{k!} + \left(\frac{(s\varrho)^s}{s!} \right) \left(\frac{1}{1-\varrho} \right) \right]^{-1}$$

Proof. The stationary distribution of P is known to be equal to

$$\pi_k = \begin{cases} \pi_0 (s\varrho)^k / k! & \text{if} \quad k \leq s \\ \pi_0 \varrho^k s^s / s! & \text{if} \quad k \geq s \end{cases}$$

where,

$$\varrho = \frac{\lambda}{s\mu} < 1$$

and,

$$\pi_0 = \left[\sum_{k=0}^{s-1} \frac{(s\varrho)^k}{k!} + \left(\frac{(s\varrho)^s}{s!} \right) \left(\frac{1}{1-\varrho} \right) \right]^{-1}$$

Hence,

$$\begin{aligned} ||\pi||_\upsilon &= \sum_{k=0}^{s-1} \beta^k \pi_k + \sum_{k=s}^{\infty} \beta^k \pi_k \\ &= \pi_0 \left(\sum_{k=0}^{s-1} \frac{(s\varrho\beta)^k}{k!} + \frac{s^s}{s!} (\varrho\beta)^s \sum_{k=0}^{\infty} (\varrho\beta)^k \right) \end{aligned}$$

We have, with assumption that $s \geq 2$, $\beta_0 = \dfrac{2\mu^2}{\lambda(\lambda+\mu)} = \dfrac{2\mu}{\lambda} \dfrac{\mu}{\lambda+\mu} < \dfrac{2\mu}{\lambda} \leq \dfrac{s\mu}{\lambda} = \dfrac{1}{\varrho}$
Then, for all $1 < \beta < \beta_0$,

$$||\pi||_\upsilon = \pi_0 \left(\sum_{k=0}^{s-1} \frac{(s\varrho\beta)^k}{k!} + \frac{(\lambda\beta/\mu)^s}{s!(1-\varrho\beta)} \right)$$

Théorème 8.4 *Let π (respectively $\widetilde{\pi}$) be the stationary distribution of the embedded Markov chain $(Q_n)_{n\geq 1}$ (respectively of the embedded Markov chain $(\widetilde{Q}_n)_{n\geq 1}$). Then, for all $1 < \beta < \beta_0$, we have :*

$$||\pi - \widetilde{\pi}||_\upsilon \leq c_0 \, c \, \Delta (1 - \rho - c\,\Delta)^{-1}$$

where c_0 is given in (8.23) , $c = 1 + ||\pi||_\upsilon$.

Proof. According to Theorem 8.2,

$$c = 1 + ||\nVdash||_{\upsilon} ||\pi||_{\upsilon},$$

where,

$$||\nVdash||_{\upsilon} = \sup_{k} \frac{1}{\beta^k} = 1.$$

Then,

$$c = 1 + ||\pi||_{\upsilon}.$$

Références

1. N. V. Kartashov (1996), *Strong Stable Markov Chains.* VSP, Utrecht, The Netherlands.
2. D. Aïssani and N. V. Kartashov, (1983). *Ergodicity and stability of Markov chains with respect to operator topologies in space of transition kernels.* Doklady Akad. Nauk AN Ukr. SSR, 11 (series A), 3-5.