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ANALYSIS OF INHOMOGENEOUS DOMAINS AND ANISOTROPIC INCLUSIONS BY THE BOUNDARY ELEMENT METHOD

Sara Yamani[1], Ahmed Sahli[1], Said Kebdani[2] and Sara Sahli[3]

[1]Université Ibn Khaldoun de Tiaret
Département de Génie Mécanique
[2]Université des Sciences et de la Technologie d'Oran (USTO MB)
E-Mail : yamanisara57@yahoo.com
[3]Université d'Oran 1
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Abstract

This paper deals with elastic 2D problems characterized by the presence of zones with different materials and anisotropic inclusions using the boundary element method. The anisotropy can be either assumed over the whole domain, or defined only over some particular inclusions, which is the most usual case. Fundamental solutions for anisotropic domains, although well-known, lead to more complex formulations and may introduce difficulties when the analysis requires more complex material models as for instance plastic behavior, finite deformations, etc. The alternative formulation proposed in this work can be applied to anisotropic bodies using the classical fundamental solutions for 2D elastic isotropic domains plus correction given by an initial stress field. The domain region with anisotropic properties or only with different isotropic elastic parameters has to be discretized into cells to allow the required corrections, while the complementary part of the body requires only boundary discretization. The initial stress tensor to be applied to the anisotropic region is defined as the isotropic material elastic stress tensor correction by introducing a local penalty matrix. This matrix is obtained by the difference between the elastic parameters between the reference values and the anisotropic material. This technique is particularly appropriate for anisotropic inclusion analysis, in which the domain discretization is required only over a small region, therefore increasing very little the number of degrees of freedom of the final algebraic system. The numerical results obtained by using the proposed

formulation have demonstrated to be very accurate in comparison with either analytical solutions or the other numerical values.

Keywords: Key words: Boundary element method, anisotropic inclusions, multi-region.

1. INTRODUCTION

In the analysis of problems involving non-homogeneous areas, various numerical techniques can be employed. In this sense, the technique of sub-regions is to consider each sub-domain individually and properly connected to the other through equations of equilibrium and compatibility of displacement imposed on the interface.

[1] Shows several practical analyses of structures foundations, where non-homogeneous regions are treated by the method of sub regions, even when different types of physical non-linearity are associated with each material.

The work of [2, 3], developed from [4] already indicated the ease with which the method could analyze areas compounds. Several numerical algorithms for the study of combinations of sub-regions, in order to expedite the solution of the system of algebraic equations of the BEM, have been proposed. One these algorithms, developed by [5], based on the elimination of blocks of zeros of the main matrix, significantly decreases the time resolution of the system.

[6] Proposes an alternative formulation in the analysis of problems of the non-homogeneous, treating the field as continuous, without the need to divide it into sub-regions, only modifying the integrals in order to take into account the differences between the elastic constants of each sub-region.

In the work of [7], the linear formulation of the BEM for bi-dimensional elasticity is used for the study of stiff areas. The consideration of the hardening is addressed in two ways, the first of the classical technique of sub-region or coupling BEM / FEM and the second also by the same coupling, but condensing variables

contour to the axis of the stiffener. The latter provided good results in eliminating disturbances hardening fine.
Although the study of problems of non-homogeneous areas, [8] presents another procedure for mounting the system of algebraic equations, very similar to that used in the finite element method, as construction of a stiffness matrix K for each region, considering factors such as streams or surface forces due to temperature / unit displacement. This method is more efficient than the method of sub-regions in the implementation in parallel computing and also can be used in the method of coupling boundary element with finite element method.
The analysis of anisotropic solid plans, having anisotropy general through BEM began with the work of [9] that, using functions of complex variables and formalism elastic anisotropic. [10], proposed a two-dimensional fundamental solution has been widely used in many different applications BEM anisotropy in general.
[11] Presented a formulation for the BEM, for analysis of anisotropic media plans, where the integral equations are discretized in the complex plane, there by differentiating it from formulations usual. The unknowns of the problem are assumed to be linear functions of a complex variable in each element boundary, and the integrations are performed accurately for arbitrary contours without the need for numerical integrations, thus constituting the advantage of the method proposed.
[12] Proposed a fundamental solution deduced from the fundamental solution of isotropic Kelvin, avoiding, this way, for three-dimensional problems, the numerical integration to determine the Green's function. The technique is to express the constants anisotropic as a mean value of the constants a more isotropic residue, which in turn is transformed into a term of the equation field text and can be treated, for example, the dual reciprocity method.
For infinite transversely isotropic media, [13] obtained a fundamental solution using three potential functions displacements. This solution was used by [14], in study of solids

subject to gravity and also by [15] in the analysis of transversely isotropic piezoelectric solids.
In the study of composite materials, we highlight the work of [16], which employs the fundamental solution in the study of orthotropic laminated anisotropic. The anisotropy, therefore, is obtained by forming the orthotropic layers in different directions with each other.
The presence of initial fields of deformation or stress applied to the area of the body is important in problems where domain variables are of importance in the mechanical problem, such as in thermo-elasticity, shrinkage and creep.
In materials that exhibit nonlinear behavior (plasticity, damage, viscous effects), the problem is solved incrementally using initial strain or stress.
In this paper we present the integral representations for problems of initial fields in the area. Subsequently, we present the approaches of the variables in the field, the use of internal nodes of the cell and the BEM algebraic equation with initial stress field.
The main objective of this work is the development of a program computational method using the boundary element in the solution of linear elastic problems flat (plate) consisting of areas no homogeneous, determining the displacements / forces on the boundary surface and tensions in the body.
The non-homogeneity in this work is restricted to problems where the domain is composed of several sub-regions, whose elastic properties of the material do not vary within each sub-region.
The classic technique of sub-regions, proposed by compliance the displacements and surface forces at the nodes of the interface allows only the study of isotropic materials. However, the formulation proposed in this paper also enables the analysis of problems with sub-regions isotropic and anisotropic. Guaranteed quality of results to the problems described above extends the formulation in the analysis of anisotropic solid media. The integral representations of initial fields in the area are used in the formulation of the problem. Therefore, it is necessary the discretization of the domain cells.

2. INTEGRAL REPRESENTATIONS OF INITIAL FIELDS

The components of the strain tensor at a point "s" are:

$$\epsilon_{ij}(s) = \epsilon_{ij}^{e}(s) + \epsilon_{ij}^{0}(s) \qquad (1)$$

Where $\epsilon_{ij}^{e}(s)$ is the elastic strain tensor and $\epsilon_{ij}^{0}(s)$ the initial strain tensor.

Equivalently, the components of the stress tensor are:

$$\sigma_{ij}(s) = \sigma_{ij}^{e}(s) - \sigma_{ij}^{0}(s) \qquad (2)$$

The tensors with initial fields preserve the elastic constitutive relation,

$$\sigma_{ij}^{0}(s) = C_{ijkl}\epsilon_{kl}^{0} \qquad (3)$$

Where C_{ijkl} is a fourth order tensor that characterizes the material, varying from point to point within the body when it is not homogeneous.

The constitutive relation can now be represented in terms of increased initial fields as follows:

$$\sigma_{ij} = \frac{2G\nu}{(1-2\nu)}\delta_{ij}\left[\varepsilon_{kk}(s) - \epsilon_{ll}^{0}(s)\right] + 2G\left[\varepsilon_{ij}(s) - \epsilon_{ij}^{0}(s)\right] \qquad (4)$$

Where G is the shear modulus and ν is the Poisson's ratio.

Similarly, the Navier's equation and surface forces also have terms referring to the initial deflection, as follows:

$$\sigma_{ij}(s) = \frac{2G\nu}{(1-2\nu)}u_{l,l}(s)\delta_{ij} + G\left[u_{i,j}(s) + u_{j,i}(s)\right] \underbrace{- \frac{2G\nu}{(1-2\nu)}\varepsilon_{kk}^{0}(s)\delta_{ij} - 2G\epsilon_{ij}^{0}(s)}_{\sigma_{ij}^{0}} \qquad (5)$$

$$p_j(S) = \frac{2G\nu}{(1-2\nu)}u_{j,j}(S)\eta_i + G\left[u_{j,i}(S)\eta_j + u_{i,\eta}(S)\right] - \frac{2G\nu}{(1-2\nu)}\varepsilon_{mm}^{0}(S)\eta_i - 2G\epsilon_{ij}^{0}(S)\eta_j \qquad (6)$$

now using the tensors ε_{ij} and σ_{ij} related to the equations (1) and (2) for the real problem in the reciprocity theorem of Betti and using the fundamental solution to Kelvin, arrives at the identity

Somigliana relating the boundary displacements $u_j(Q)$ with the boundary tractions $P_j(Q)$ plus the terms of initial tension,

$$c_{ij}(s)u_j(s) = \int_{\Gamma} P_j(Q)u^*_{ij}(s,Q)d\Gamma - \int_{\Gamma} P^*_{ij}(s,Q)u_j(Q)d\Gamma + \int_{\Omega} b_j(q)u^*_{ij}(s,q)d\Omega + \int_{\Omega} \sigma^0_{jk}(q)\varepsilon^*_{ik}(s,q)d\Omega \qquad (7)$$

Where $c_{ij}(s)$ is generally a function of the geometry variation at the boundary point s. Providing that s is a smooth boundary point, that is, the outward normal vector to the boundary is continuous at s, then it can be shown that $C_{ij}(s)=1/2\delta_{ij}$ [17]. $P^*_{ij}(s, Q)$ and $u^*_{ij}(s,Q)$ represent the traction and displacement fundamental solutions at a boundary point Q due to a unit load placed at location s.

3. APPROXIMATION FUNCTIONS OF VARIABLES IN THE DOMAIN

As has been discussed, the consideration of initial fields introduces the integral domain in addressing the problem. The simplest way to calculate such integrals is turning them into summations over domain discretization units, or cells.

In this section, triangular cells are used and all the stress equations are written for points belonging to the area. That is, all nodes of the cells do not coincide with their vertices (or nodes geometry) as they are pulled into the domain of the cell, passing a field belonging to the body.

The domain Ω is discretized into N_c cells, and the stress components are approximated by functions polynomial of the form:

$$^m\sigma^0_{ij}(s) = \phi_k(s)\,^m\sigma^{0k}_{ij} \qquad (15)$$

Where m - represents the cell Ω_m.
k - represents the nodes of the cell.

${}^{m}\sigma^{0k}{}_{ij}$ - represents the variable component *ij* of the nodal tensor of initial tractions of the cell m to node *k*
In this work, the polynomial $\phi_k(s)$ are adopted linear.
For the integrals over the cells, the numerical integration in two dimensions can be avoided if a semi-analytical procedure is adopted, or is, the first integral in the variable radius is calculated analytically and then the variable angle is evaluated numerically.
Two domain integrals should be calculated in the formulation proposed in this paper. The first kernel ε^*_{ijk} from the equation of displacement (7) has weak singularity (1/r) and the second kernel E^*_{ijkl} from the traction equation (11) presents a strong singularity ($1/r^2$).

4. EXAMPLES

This section presents some examples of application of the proposed formulation and compares results with those obtained with the finite element method with analytical solutions (when available) and / or other results in the literature.
The computer program developed based on the formulation proposed in this work was employed analytical integrations over the boundary element and numerical integrations over the elements in the field. In all the examples presented, we adopted yet, the value of dist = 0.40, dimensionless distance, as shown in Figure 1.

4-1 Thick wall tube

This example is the analysis of a thick piece tubular, formed by two concentric tubes properly engaged. The first tube has a thickness and inner radius equal to 2. The outer tube has the same thickness as the interior and its elasticity module is twice of the first tube.
The first analysis consists in examining the convergence of the solution, comparing the results in radial displacements and stresses with the analytical solution of the problem available in the literature. The geometry, material properties and boundary conditions imposed are shown in Figure 2. It is important to note

that results presented below refer to the State plane stress. 3 discretizations were considered, of which the first with 6 linear elements on the inner tube of the quarter under review, 14 linear elements on the outside, 4 linear elements in each line of symmetry and 67 cells to discretize the continuous medium penalized.

The second mesh is the refinement of the first model using 12 linear elements on the inner surface of the quarter tube, 28 linear elements on the outside, 8 linear elements in each line of symmetry and 268 cells to discretize the continuous medium penalized.

The final discretization consists of 20 linear elements on its inner surface 40 linear elements on the outside, 16 linear elements each line of symmetry and 692 cells to discretize the continuous medium penalized.

The following radial displacement for the 5 points above was found and is shown in Figures 3 to 7:

The average error quadratic obtained considering all the radial displacement of the nodes in the first contour discretization was 2.73%, decreasing to 0.72% in the second discretization and closing at 0.49% last discretization.

4-2 Beam

The loading, geometry, and boundary conditions of the problem are seen in the figure 8. 2 discretizations were adopted, the first of which consists of 70 elements in the contour and 120 cells

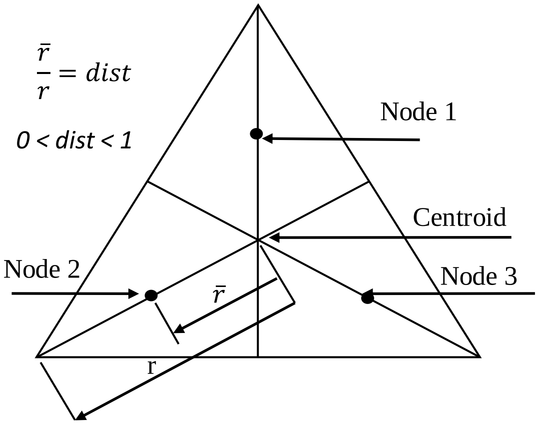


Figure 1: Position of the cell internal nodes

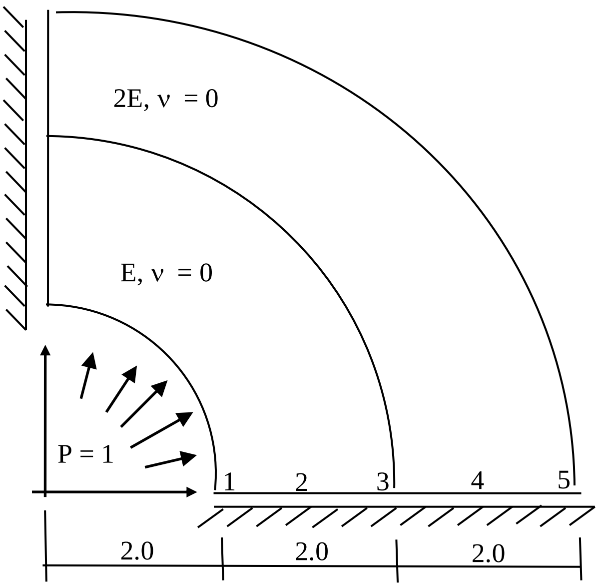


Figure 2: Thick wall tube. Geometry and loading

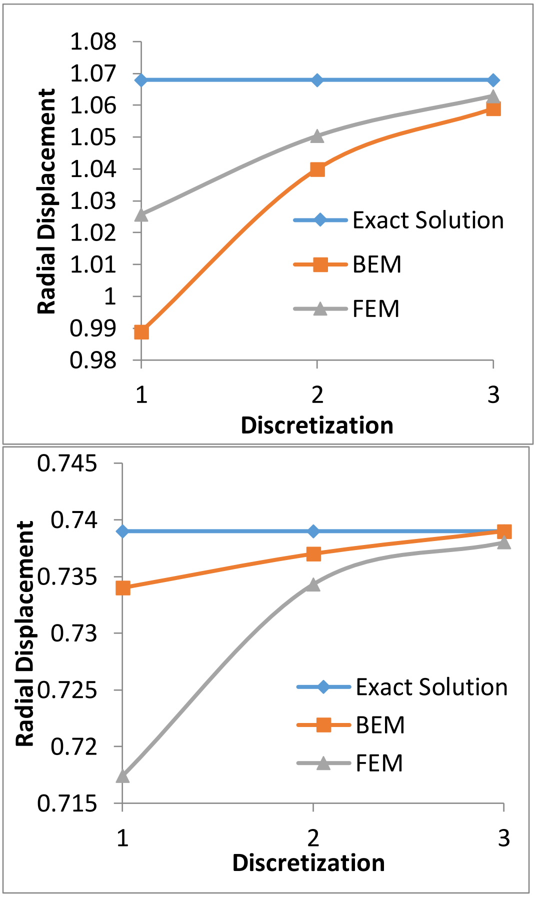


Figure 3-4: Radial displacement of point 1 and 2 respectively in 3 discretizations analyzed

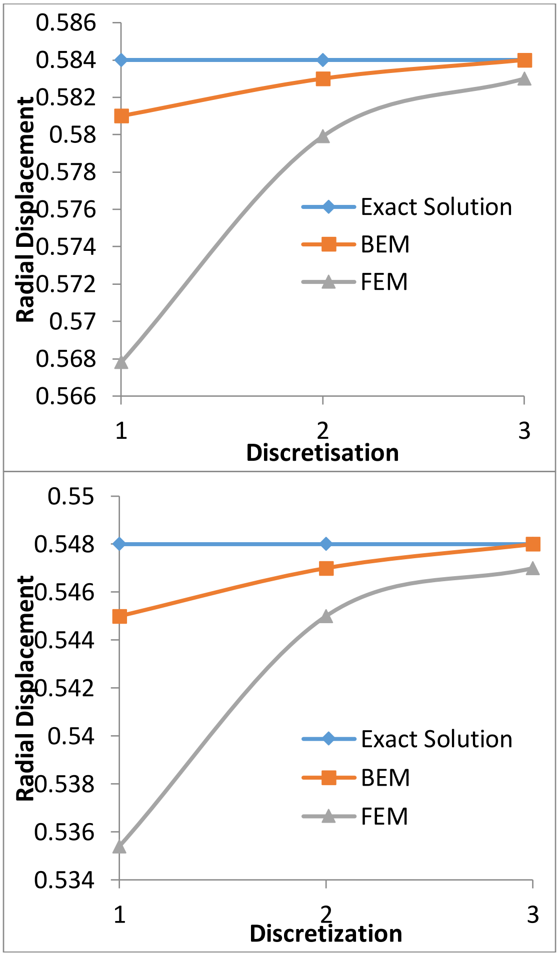


Figure 5-6: Radial displacement of point 3 and 4 respectively in 3 discretizations analyzed

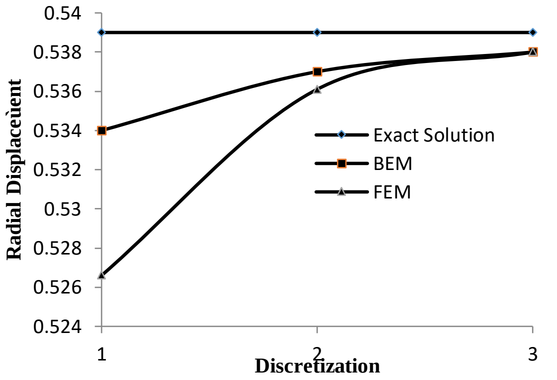


Figure 7: Radial displacement of point 5 in 3 discretizations analyzed

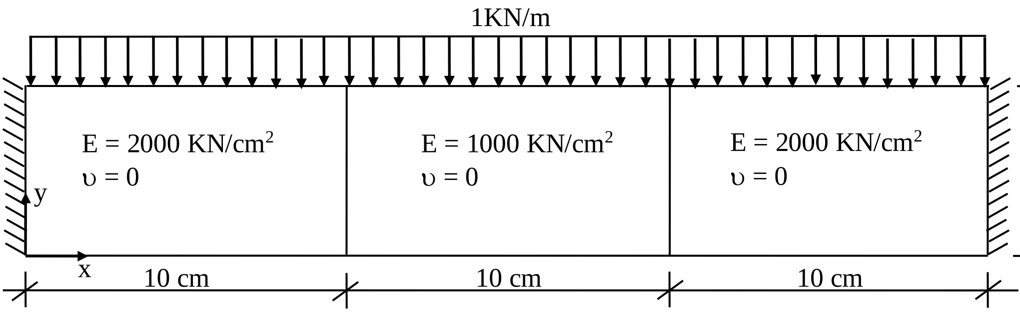
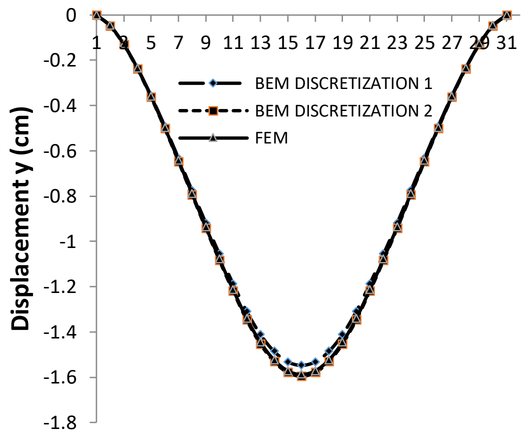


Figure8: Geometry and loading



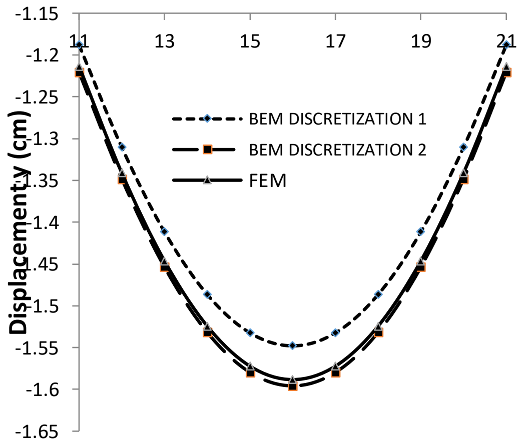


Figure 9: vertical displacements along the Figure 10: vertical displacements only in the central underside of the beam. beam.

in the domain, in the second discretization, 140 were adopted on the boundary elements and 480 cells in the domain.
The results in vertical displacement at the nodes on the underside of the beam are presented in the figures below, comparing the two discretizations adopted with the results obtained by the FEM based on the same discretization of the domain.

In Figures 9 and 10 are presented the vertical displacements that occur along the underside of the beam.

5. CONCLUSION

The aim of this paper is to present an alternative formulation to solve problems of the sub region or anisotropic inclusion, through BEM modifying the elastic parameters with initial stress field. In particular, developed a computer program based on the formulation, which was subjected to a series of examples and their results were compared to those obtained by other techniques.
Because of the difficulty of getting the 21 constants needed to characterize a material with complete anisotropy, were analyzed, at most, orthotropic material properties.
The accuracy of the results demonstrates the feasibility of using this formulation to the analysis of problems where one party or the domain for complete, is presented with anisotropic properties. The use of cells with discontinuity in the variables of initial stress demand a large area of computer memory for the assembly and resolution of the algebraic system. Thus, the formulation is more suitable for problems with inclusions that require discretization in regions smaller.

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