Pseudo-differential operators and applications to Partial differential equations

ELONG Ouissam [1], **SENOUSSAOUI Abderrahmane** [2]
Laboratoire de Mathématiques Fondamentales et Appliquées d'Oran LMFAO
elongouissam@yahoo.fr [1], senoussaoui_abdou@yahoo.fr

Abstract: **The aim of this work is to study pseudo-differential operators. We define this operator's class and give an application to the theory of partial differential equations.**

Keyword: **Pseudo-differential operators, symbols, elliptic PDO.**

I. Introduction

The theory of pseudo-differential operators (which in what follows will be abbreviated as PDO) is a tool to solve elliptic partial differential equations. It is developed in the middle of sixty thank's to works of Hörmander, Nirenberg and others. This theory has a great role in mathematics and mathematics-physic like microlocal analysis and quantum physics.

The general form of pseudo-differential operator I is

$$(Iu)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i<x-y,\xi>} a(x,y,\xi) u(y) dy \, d\xi, \tag{1}$$

where a is a symbol and $u \in C_0^\infty(\mathbb{R}^n)$ (the space of infinitely differentiable functions with compact support).

A linear partial differential operator P of order m is an application in $C^\infty(\mathbb{R}^n)$

$$(Pu)(x) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha u(x),$$

with $a_\alpha \in C^\infty(\mathbb{R}^n)$ are the coefficients of the operator P.

The Fourier transform of an application u is defined by

$$\widehat{u}(\xi) = (\mathcal{F}u)(\xi) = \int_{\mathbb{R}^n} e^{-i<x,\xi>} u(x) dx$$

is an isomorphism in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ with inverse given by

$$(\mathcal{F}^{-1}v)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i<x,\xi>} v(\xi) d\xi.$$

The Fourier transform of an application converts differential problems to algebraic problems, one of its elementary properties is

$$\widehat{D^\alpha u}(\xi) = \xi^\alpha \widehat{u}(\xi)$$

From the previous formulas we deduce

$$\begin{aligned} D^\alpha u(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i<x,\xi>} \xi^\alpha \widehat{u}(\xi) d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i<x-y,\xi>} \xi^\alpha u(y) dy \, d\xi \end{aligned}$$

So we can write P as

$$(Pu)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i<x-y,\xi>} p(x,\xi) u(y) dy \, d\xi,$$

where p is the *symbol* of the differential operator P defined by

$$p(x,\xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha,$$

is a polynomial of degree m in ξ with coefficients depending in x.

So we have represented a differential operator by its symbol using Fourier transform and its basic properties. We can extend this representation by considering a new class of symbols that are not polynomials in ξ. This new operators are called "Pseudo-differential operators".

II. Pseudo-differential operators

A. Notions on PDO

In the sequel, we assume X an open set in $\mathbb{R}^n$. A PDO is defined by its symbol.

***Definition** 2.1: Let $m, \rho, \delta \in \mathbb{R}$ with $0 \leq \rho, \delta \leq 1$. A symbol is a function $a \in C^\infty(X \times \mathbb{R}^n)$ such that for any compact $K \subset X$ and any multi-indices α and β, there exists a constant $C_{\alpha,\beta,K}$ for which*

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leq C_{\alpha,\beta,K}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|},$$

where $x \in K$, $\xi \in \mathbb{R}^n$.

We note $S^m_{\rho,\delta}(X \times \mathbb{R}^n)$ the set of symbols of order am and type ρ, δ and $S^{-\infty}_{\rho,\delta}(X \times \mathbb{R}^n) = \bigcap_m S^m_{\rho,\delta}(X \times \mathbb{R}^n)$.

***Example** 2.1:* If $a \in C^\infty$ and a is a homogeneous function of degree m with respect to ξ for large $|\xi|$, then a is a symbol of degree m and type $1, 0$.

To facilitate calculus on symbols we mention some elementary properties.

***Proposition** 2.1:*
(i) *If $a \in S^m_{\rho,\delta}$ then $\partial_x^\alpha \partial_\xi^\beta a \in S^{m-\rho|\alpha|+\delta|\beta|}$;*
(ii) *If $m \leq l$ then $S^m_{\rho,\delta} \subset S^l_{\rho,\delta}$;*
(iii) *If $a \in S^m_{\rho,\delta}$ et $b \in S^l_{\rho,\delta}$, then $ab \in S^{m+l}$.*

***Definition** 2.2: The integral*

$$(Iu)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i<x-y,\xi>} a(x,y,\xi) u(y) dy\, d\xi,$$

where $a \in S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n)$ and $u \in C_0^\infty(\mathbb{R}^n)$ is said pseudo-differential operator.

The set of PDO is denoted $\Psi^m_{\rho,\delta}(X)$.

***Definition** 2.3: The distribution $K \in \mathcal{D}'(X \times X)$ defined by*

$$K(x,y) = \int e^{i(x-y)\xi}\, a(x,y,\xi)\, d\xi \tag{2}$$

is called kernel of the PDO I.

***Example** 2.2:* A linear partial differential operator of order m

$$A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

where $a_\alpha \in C^\infty$ defines a PDO of order m and type $1, 0$, his symbol is given by

$$\sigma_A(x,\xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha.$$

To give a meaning to the previous integral we introduce the notion of phase function and oscillatory integrals.

***Definition** 2.4: We call $\phi(x,\xi)$ a phase function if $\phi(x,\xi) \in C^\infty(X, \mathbb{R}^n \setminus 0)$, $\phi(x,\xi)$ is real valued and positively homogeneous of degree 1 in ξ (i.e. $\phi(x,t\xi) = t\phi(x,\xi)$ for any $x \in X$, $\xi \in \mathbb{R}^n \setminus 0$ and $t > 0$) and $\phi(x,\xi)$ does not have critical points for $\xi \neq 0$.*

***Definition** 2.5: The integral*

$$(Iu)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi) u(x)\, dx\, d\xi$$

in which $a(x,\xi) \in S^m_{\rho,\delta}(X \times \mathbb{R}^n)$ and $\phi(x,\xi)$ is a phase function is called an oscillatory integral.

The integral just defined converges if $m < -n$. In the other case we use the technique of oscillatory integrals developed by Hörmander which consists to construct a differential operator L that satisfies

$${}^tLe^{i\phi(x,\xi)} = e^{i\phi(x,\xi)},$$

where ${}^tL$ is the transpose of the differential operator L. Using this fact, an integration by parts k times shows that

$$(Iu)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} L^k(a(x,\xi)u(x))\, dx\, d\xi \tag{3}$$

Putting $s = \min(\rho, 1-\delta)$, we deduce $L^k(au) \in S^{m-ks}_{\rho,\delta}(X \times \mathbb{R}^n)$. If $\rho > 0$ and $\delta < 1$ (so that $s > 0$), which will always be assumed in the sequel, then the formula (3) allows us to define the integral I for an arbitrary m if we select k so that $m - ks < -n$. This makes the integral (3) absolutely convergent.

***Proposition** 2.2: A is a linear bounded operator from $C_0^\infty(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$.*

We study now the algebraic structure of the set of PDO. To this end if we want to calculate the product (composition) of two PDO A and B, one has to require that the image of the operator B is compact, if we consider $A \circ B$, but in general we have not this case, so we have to introduce another class of pseudo-differential operators known as " Properly supported pseudo-differential operators".

B. The Algebra of Pseudo-differential operators

Let I be a PDO with kernel K_I and let $supp_{K_I}$ denotes the support of K_I. Recall that a continuous map $f : X \to Y$ between topological spaces X and Y is called *proper* if for any compact $K \subset Y$, the inverse image $f^{-1}(K)$ is a compact in X.

***Definition** 2.6: A PDO I is called properly supported if both canonical projections $\Pi_1, \Pi_2 : supp_{K_I} \to X$ are proper maps.*

So we can state the following theorem.

***Theorem** 2.1: Let I be a properly supported PDO. Then I defines a map*

$$I : C_0^\infty(X) \to C_0^\infty(X).$$

To give some combination between classes of PDO we give the important theorem.

***Theorem** 2.2: Any pseudo-differential operator I can be written in the form $I = I_0 + I_1$ where I_0 is a properly supported PDO and I_1 has kernel $K_{I_1} \in C^\infty(X \times X)$.*

We return now to symbol calculus by introducing the symbol of properly supported PDO.

***Definition** 2.7: Let I be a properly supported PDO. Its symbol is a function $\sigma_I(x, \xi)$ on $X \times \mathbb{R}^n$ defined by*

$$\sigma_I(x, \xi) = e_{-\xi}(x)\, I\, e_\xi(x),$$

where $e_\xi(x) = e^{ix\xi}$.

Since $e^{ix\xi}$ is an infinitely differentiable function of ξ with values in $C^\infty(X)$ and I is a continuous linear operator on $C^\infty(X)$, it is clear that $\sigma_I(x, \xi)$ is also an infinitely differentiable function of ξ taking values in $C^\infty(X)$, therefore $\sigma_I(x, \xi) \in C^\infty(X \times \mathbb{R}^n)$.

*1) **An expression for the symbol of a properly supported PDO**:* In this and the following paragraphs we assume $\delta < \rho$.

***Theorem** 2.3:* Let I be a properly supported PDO and $\sigma_I(x, \xi)$ its symbol. Then

$$\sigma_I(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha\, D_y^\alpha\, a(x, y, \xi)|_{y=x}.$$

*2) **The symbol of the transposed operator**:* The transposed operator ${}^tI$ is defined by

$$< Iu, v > = < u, {}^t Iv >$$

for any $u, v \in C_0^\infty(X)$, where

$$< u, v > = \int u(x)v(x)dx.$$

Therefore, if $I \in \Psi_{\rho,\delta}^m(X)$ where $a \in S_{\rho,\delta}^m(X \times X \times \mathbb{R}^n)$, the transpose ${}^tI$ is given by

$${}^tIv(y) = \iint e^{i(x-y)\xi}\, a(x, y, \xi)v(x)dx\, d\xi;$$

which with the change of variable $\eta = -\xi$ gives

$${}^tIv(y) = \iint e^{i(y-x)\xi}\, a(x, y, -\eta)v(x)dx\, d\eta.$$

Consequently, $t^I \in \Psi_{\rho,\delta}^m(X)$.

***Theorem** 2.4: Let I be a properly supported PDO with symbol $\sigma_I(x, \xi)$ and $\sigma_I'(x, \xi)$ the symbol of ${}^tI$, then*

$$\sigma_I'(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha\, D_x^\alpha \sigma_I(x, -\xi).$$

*3) **The composition formula**:*

***Theorem** 2.5: Let I_2 and I_2 be two properly supported PDO in X and let their symbols be $\sigma_{I_1}(x, \xi)$ and $\sigma_{I_2}(x, \xi)$ respectively. Then the composition BA is then a properly supported PDO, whose symbol satisfies the relation*

$$\sigma_{BA}(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_{I_2}(x, \xi) D_x^\alpha \sigma_{I_1}(x, \xi).$$

C. Boundedness and compactness of PDO

Operator's continuity is an important question often asked in the theory of operators. When the operator on question is linear, the notions of continuity and boundedness are equivalents. So we are interesting to study its boundedness between Banach spaces and particularly the famous Lebegue's spaces L^2. The importance of the space L^2 is arisen from its structure, it is complete, defines a Hilbert space and in addition is the most space used by physicists. So in this paragraph we treat L^2 boundedness and L^2 compactness of PDO.

***Theorem** 2.6: Let $I \in \Psi^0_{\rho,\delta}(\mathbb{R}^n)$, $0 \leq \delta < \rho \leq 1$ and $supp_{K_I}$ be compact in $\mathbb{R}^n \times \mathbb{R}^n$. Then*

$$||Iu|| \leq C||u||$$

where $C > 0$, and I can be extended to a linear continuous operator on $L^2(\mathbb{R}^n)$.

For the proof we need the following theorem.

***Theorem** 2.7: Let I be a properly supported PDO in $\Psi^0_{\rho,\delta}(\mathbb{R}^n)$, with $0 \leq \delta < \rho \leq 1$ and X an open set in $\mathbb{R}^n$. Suppose there exists a constant M such that*

$$\overline{\lim_{\substack{|\xi|\to\infty \\ x\in K}}} |\sigma_I(x,\xi)| < M,$$

for any compact set $K \subset X$. Then there exists a properly supported integral operator with hermitian kernel $R \in C^\infty(X \times X)$ such that

$$(Iu, Iu) \leq M^2(u,u) + (Ru,u), \ \forall u \in C_0^\infty(X).$$

If, in addition, $supp_{K_I}$ is compact in $X \times X$ then $supp\, R$ is also compact in $X \times X$.

We deduce the compactness theorem.

***Theorem** 2.8: Let $I \in \Psi^0_{\rho,\delta}(\mathbb{R}^n)$ with $0 \leq \delta < \rho \leq 1$ let $supp_{K_I}$ be compact in $\mathbb{R}^n \times \mathbb{R}^n$ and*

$$\sup_x |\sigma_I(x,\xi)| \to 0 \ \text{ as } |\xi| \to +\infty.$$

Then I extends to a compact operator in $L^2(\mathbb{R}^n)$.

III. Applications

The main question in the theory of partial differential equations is how to solve the equation

$$Lu = f$$

for a given partial differential operator L and a given function f . In other words, how to find the inverse of L, i.e. an operator L^{-1} such that

$$L \circ L^{-1} = L^{-1} \circ L = I \tag{4}$$

is the identity operator (on some space of functions where everything is well-defined). In this case function $u = L^{-1}f$ gives a solution to the partial differential equation $Lu = f$. First of all we can observe that if operator L is an operator with variables coefficients in most cases it is impossible or is very hard to find an explicit formula for its inverse L^{-1} (even when it exists). However, in many questions in the theory of partial differential equations one is actually not so much interested in having a precise explicit formula for L^{-1}. Indeed, in reality one is mostly interested not in knowing the solution u to the equation $Lu = f$ explicitly but rather in knowing some fundamental properties of u. Thus, the question becomes whether we can say something about singularities of u knowing singularities of $f = Lu$. In this case we do not need to solve equation $Lu = f$ exactly but it is sufficient to know its solution modulo the class of smooth functions. Namely, instead of L^{-1} in (4) one is interested in finding an "approximate" inverse of L modulo smooth functions, i.e. an operator P such that

$$u = Pf$$

solves equation $Lu = f$ modulo smooth functions, i.e. if $(PL - I)f$ and $(LP - I)f$ are smooth for all functions f from some class.

***Definition** 3.1: A symbol $a \in S^m_{1,0}(X \times \mathbb{R}^n)$, $m \in \mathbb{R}$, is said to be elliptic if there exist $C, R > 0$ such that*

$$|a(x,\xi)| \geq C|\xi|^m, \qquad \forall |\xi| \geq R, \ x \in X.$$

***Definition** 3.2: A pseudo-differential operator is called elliptic if its symbol is elliptic.*

***Example** 3.1:* The symbol of the Laplacian operator $\Delta = \sum_{j=1}^{n} D_{x_j}^2$ is $-\sum_{j=1}^{n} \xi^2$, so it is elliptic.

***Definition** 3.3: Let $A \in \Psi_{1,0}^m(\mathbb{R}^n)$. A properly supported pseudo-differential operator P is called parametrix of A if it satisfies*

$$PA = I + R_l \text{ and } AP = I + R_r,$$

where $R_l, R_r \in \Psi_{1,0}^{-\infty}(\mathbb{R}^n)$.

***Theorem** 3.1: Any elliptic operator $A \in \Psi_{1,0}^m(\mathbb{R}^n)$ has a parametrix $P \in \Psi_{1,0}^{-m}(\mathbb{R}^n)$.*

IV. FOURIER INTEGRAL OPERATORS

Another family of operators intimately connected to the theory of PDO is known as Fourier integral operators which has the form

$$Fu(x) = \int e^{i\phi(x,\theta)} a(x,\theta) \mathcal{F}u(\theta)\, d\theta$$

where $u \in \mathcal{S}(\mathbb{R}^n)$, $a(x,\theta)$ is a symbol and $\phi(x,\theta)$ is a phase function.

So a PDO is a Fourier integral operator with phase function $< x, \theta >$.

This class of operators appear in the expression of the solutions of the hyperbolic partial differential equations and is a generalization.

V. CONCLUSION AND PERSPECTIVES

In this article we defined PDO and studied most important properties and different calculus on its. We treated continuity in some classes of symbols by given some conditions on types of symbols. Our purpose is to generalize this results by considering another types of symbols and conditions on the phase functions in linear and multi-linear cases for both pseudo-differential and Fourier Integral operators.

REFERENCES

[1] H. Abels. *Pseudodifferential and singular integral operators. An introduction with applications.* Berlin: de Gruyter, 2012.

[2] S. Alinhac , P. Gérard. *Opérateurs pseudo-différentiels et théorème de Nash-Moser.* Providence, RI: American Mathematical Society (AMS), 2007.

[3] L. Hörmander. Fourier integral operators. I. *Acta Math.*, 127: 79–183, 1971.

[4] L. Hörmander. *The analysis of linear partial differential operators. III: Pseudo-differential operators. Reprint of the 1994 ed.* Berlin: Springer,Reprint of the 1994 ed. 2007.

[5] B. Messirdi, A. Senoussaoui. *L^2 boundedness and L^2 compactness of a class of Fourier integral operators.* Electronic Journal of Differential Equations, Vol 2006 (26) (2006) 112.

[6] S. Rodriguez-Lopez; W. Staubach. *Estimates for rough Fourier integral and pseudodifferential operators and applications to the boundedness of multilinear operators.* Journal of Functional Analysis 264 (2013), 2356-2385.

[7] M. Ruzhansky. *Introduction to pseudo-differential operators* Lecture notes. Imperial College London, UK, 2014.

[8] X. Saint Raymond. *Elementary introduction to the theory of pseudodifferential operators.* Boca Raton, FL: CRC Press, 1991.

[9] M. A. Shubin. *Pseudodifferential operators and spectral theory. Transl. from the Russian by Stig I. Andersson. 2nd ed.* Berlin: Springer, 2nd ed.edition, 2001.

[10] M. Taylor. *Tools for PDE.* AMS, Providence RI, 2000.