$\mathbb{Z}_4\mathbb{Z}_4[u^3=1]-$Cyclic codes and their reversible codes

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***Abstract*—Recently some special type of mixed alphabet codes that generalize the standard codes has attracted much attention. Besides $\mathbb{Z}_2\mathbb{Z}_4-$additive codes, $\mathbb{Z}_2\mathbb{Z}_2[u]-$linear codes are introduced as a new member of such families. In this paper, we are interested in a new family of such mixed alphabet codes, i.e., codes over $\mathbb{Z}_4\mathbb{Z}_4[u]$ with $u^3=1$, we study the structure of cyclic codes over the ring $\mathbb{Z}_4R=\mathbb{Z}_4(\mathbb{Z}_4+u\mathbb{Z}_4+u^2\mathbb{Z}_4)$ with $u^3=1$. The reversible cyclic codes of arbitrary length over $\mathbb{Z}_4R$ are discussed. It is worth noting that the $\mathbb{Z}_4-$Gray images are $\mathbb{Z}_4-$linear codes.**

***Index Terms*—Cyclic codes, Gray map, reversible codes.**

I. Introduction

Recently, inspired by the $\mathbb{Z}_2\mathbb{Z}_4-$additive codes (introduced in [3]), $\mathbb{Z}_2\mathbb{Z}_2[u]-$linear codes have been introduced in [2]. Though these code families are similar to each other, $\mathbb{Z}_2\mathbb{Z}_2[u]-$linear codes have some advantages compared to $\mathbb{Z}_2\mathbb{Z}_4-$additive codes. For example, the Gray image of any $\mathbb{Z}_2\mathbb{Z}_2[u]-$linear code will always be a linear binary code. This property does not hold for $\mathbb{Z}_2\mathbb{Z}_4-$additive codes.

In 2007, Siap and Abualrub [1] studied the structure of reversible cyclic codes over $\mathbb{Z}_4$. In 2015, Srinivasulu and Bhaintwal [7] studied reversible cyclic codes over $\mathbb{F}_4+u\mathbb{F}_4, u^2=0$ and their applications to DNA codes, Sehmi *et al.* [6] studied reversible and reversible complement cyclic codes over Galois rings. Motivated by these works, we study reversible cyclic codes of arbitrary length n over $\mathbb{Z}_4(\mathbb{Z}_4+u\mathbb{Z}_4+u^2\mathbb{Z}_4)$ with $u^3=1$. Recall that these codes have applications in DNA computing which is a field of study that aims at harnessing individual molecules at the nanoscopic level for computational purposes. Computation with DNA molecules possesses an inherent interest for researchers in computer and biology. At present, many researchers have been interested in designing a new set of codewords for each experiment depending on various design constraints in DNA computing. One can prevent errors by minimizing the similarity between the sequences under some distance measure. These codes have many applications in constructing data storage and retrieval systems.

The paper is organized as follows: In Section 2, we give some basic definitions. In Section 3, the structure of cyclic codes of arbitrary length over the ring $\mathbb{Z}_4R$ is determined. In Section 4, we study reversible cyclic codes of arbitrary length over $\mathbb{Z}_4R$.

II. Preliminaries.

Let R the commutative, characteristic 4 ring $\mathbb{Z}_4+u\mathbb{Z}_4+u^2\mathbb{Z}_4=\{a+ub+u^2c \mid a,b,c\in\mathbb{Z}_4\}$, with $u^3=1$. This ring can be written as the quotient ring $\mathbb{Z}_4[u]/\langle u^3-1\rangle$. This ring is a non finite chain ring.

A linear code C of length n over R is an $R-$submodule of R^n. An element of C is called a codeword. A code C is said to be cyclic, if C is closed under the cyclic shift $\rho: R^n\to R^n$, defined by $\rho(a_0,a_1,\ldots,a_{n-1})=(a_{n-1},a_0,\ldots,a_{n-2})$. It is well known that C is a cyclic code of length n over R if and only if C is an ideal of the ambient ring $R_n=R[x]/\langle x^n-1\rangle$.

Let $l\in R$. Then l can be expressed in the form $l=a+ub+u^2c$, where $a,b,c\in\mathbb{Z}_4$. Define the following map

$$\begin{array}{c}\eta: R\to\mathbb{Z}_4\\ r=a+ub+u^2c\mapsto b.\end{array} \tag{1}$$

It is clear that the mapping η is a ring homomorphism.

For any $l\in R$ and $x\in\mathbb{Z}_4R$, we define the following $R-$scalar multiplication on $\mathbb{Z}_4R$ as $R\times\mathbb{Z}_4R\to\mathbb{Z}_4R$ such that $l\star(e\mid r)=(\eta(l)e\mid lr)$. This is a well-defined multiplication. It can be extended component-wise over $\mathbb{Z}_4^\alpha\times R^\beta$ as follows: $R\times\mathbb{Z}_4^\alpha\times R^\beta\to\mathbb{Z}_4^\alpha\times R^\beta$ where

$$l\star x=(\eta(l)e_0,\eta(l)e_1,\ldots,\eta(l)e_{\alpha-1}\mid lr_0,lr_1,\ldots,lr_{\beta-1}).$$

where $x=(e_0,e_1,\ldots,e_{\alpha-1}\mid r_0,r_1,\ldots,r_{\beta-1})\in\mathbb{Z}_4^\alpha R^\beta$. By this multiplication, $\mathbb{Z}_4^\alpha R^\beta$ forms an $R-$module.

Definition 1: A non-empty subset C of $\mathbb{Z}_4^\alpha R^\beta$ is called an $\mathbb{Z}_4R$-linear code of length (α,β) if C is an $R-$submodule of $\mathbb{Z}_4^\alpha R^\beta$.

Let C be a $\mathbb{Z}_4R$-linear code and let C_α (respectively C_β) be the canonical projection of C on the first α (respectively on the last β) coordinates. Since the canonical projection is a linear map, C_α and C_β are linear codes over $\mathbb{Z}_4$ and over R of length α and β, respectively. A code C is called separable if C is the direct product of C_α and C_β, i.e.,

$$C=C_\alpha\times C_\beta.$$

We introduce the inner product on $\mathbb{Z}_4^\alpha R^\beta$. For any two vectors

$$\mathbf{t} = (e_0, \ldots, e_{\alpha-1} \mid r_0, \ldots, r_{\beta-1}),$$

$$\mathbf{t}' = (e'_0, \ldots, e'_{\alpha-1} \mid r'_0, \ldots, r'_{\beta-1}) \in \mathbb{Z}_4^\alpha \times R^\beta$$

let

$$\langle \mathbf{t}, \mathbf{t}' \rangle = u^2 \sum_{i=0}^{\alpha-1} e_i \acute{e}_i + \sum_{j=0}^{\beta-1} r_j \acute{r}_j.$$

Definition 2: Let $x = (x_0, x_1, \ldots, x_{n-1}) \in R^n$ be an $n-$tupple. The reverse of x is defined as $x^r = (x_{n-1}, x_{n-2}, \ldots, x_1, x_0)$. Than, a linear code C_n of length n over R is said to be **reversible** if $x^r \in C_n$.
Let $f(x) = c_0 + c_1 x + \cdots + c_t x^t$ with $c_t \neq 0$, and the reciprocal polynomial of $f(x)$ is defined as $f^*(x) = x^t f(1/x) = c_t + c_{t-1}x + \cdots + c_0 x^t$. The polynomial $f(x)$ is self-reciprocal if $f^*(x) = f(x)$.

We define a Gray map as $\Phi : \mathbb{Z}_4^\alpha \times R^\beta \mapsto \mathbb{Z}_4^{\alpha+2\beta}$ such that $\Phi(x) = \Phi(e \mid r) = (e \mid \phi(r))$, where ϕ is a Gray map defined by

$$\phi : R \to \mathbb{Z}_4^2$$

$$\phi(a + ub + u^2 c) = (a + b + c, a + b, a + c),$$

The image $\mathcal{C} = \Phi(C)$ of a $\mathbb{Z}_4\mathbb{Z}_4[u]-$linear code C of length (α, β) is a linear code of length $n = \alpha + 2\beta$ over $\mathbb{Z}_4$.

The Lee weight of an element $x = (e \mid r) \in \mathbb{Z}_4^\alpha R^\beta$, where $(e_0, e_1, \ldots, e_{\alpha-1}) \in \mathbb{Z}_4^\alpha$ and $(r_0, r_1, \ldots, r_{\beta-1}) \in R^\beta$ is defined as

$$w_L(x) = w_H(\Phi(x)),$$

where w_H denotes the Hamming weight and the Lee distance between two vectors $x, y \in \mathbb{Z}_4^\alpha \times R^\beta$ is defined as $d_L(x, y) = w_L(x - y)$.

Now, we provide some results which will be useful later

Theorem 1: [1] Let C be a cyclic code in $R_\alpha = \mathbb{Z}_4[x]/\langle x^\alpha - 1\rangle$.

(i) If n is odd, then R_α is a principal ideal ring and $C = \langle f_1(x), 2f_2(x)\rangle = \langle f_1(x) + 2f_2(x)\rangle$ such that $f_2(x) \mid f_1(x) \mid (x^\alpha - 1) \bmod 4$.

(ii) If α is even number, then

a) C is a free module of generator $C = \langle f_1(x) + 2p(x)\rangle$, where $f_1(x)$ and $p(x)$ are polynomials with $f_1(x) \mid (x^\alpha - 1) mod\ 2$, and $f_1(x) + 2p(x) \mid (x^\alpha - 1) mod\ 4$,

b) $C = \langle f_1(x) + 2p(x), 2f_2(x)\rangle$, where $f_1(x), f_2(x)$ and $p(x)$ are polynomials with $f_2(x) \mid f_1(x) \mid (x^\alpha - 1) mod\ 2, f_2(x) \mid p(x)((x^\alpha - 1)/f_1(x)) mod\ 2$ and deg $f_1(x) >$ deg $f_2(x) >$ deg $p(x)$.

Abualrub and Siap discussed the reversible cyclic codes of arbitrary length over $\mathbb{Z}_4$ [1]. We summarized these convenient results for our purposes.

Theorem 2: [1] Let $C_\alpha = \langle f_1(x), 2f_2(x)\rangle = \langle f_1(x) + 2f_2(x)\rangle$ be a linear cyclic code of odd length α over $\mathbb{Z}_4$. Then C_α is reversible cyclic code if and only if $f_1(x)$ and $f_2(x)$ are self-reciprocals.

III. Cyclic codes over $\mathbb{Z}_4 R$

The purpose of this section is to determine the structure of cyclic codes over the ring $\mathbb{Z}_4 R$ of arbitrary length.

Definition 3: Let A and B be two linear codes. Then the operations $\otimes$ and $\oplus$ are defined by

$$\begin{aligned} A \otimes B \otimes C &= \{(a, b, c) : a \in A, b \in B, c \in C\} \\ A \oplus B \oplus C &= \{a + b + c : a \in A, b \in B, c \in C\}. \end{aligned} \tag{2}$$

Let C_β be a linear code over R. Then we define:

$$\begin{aligned} C_{\beta,1} &= \{x \in \mathbb{Z}_4^\beta : \exists y, z \in \mathbb{Z}_4^\beta, \epsilon_1 x + \epsilon_2 y + \epsilon_3 z \in C_\beta\} \\ C_{\beta,2} &= \{y \in \mathbb{Z}_4^\beta : \exists x, z \in \mathbb{Z}_4^\beta, \epsilon_1 x + \epsilon_2 y + \epsilon_3 z \in C_\beta\} \\ C_{\beta,3} &= \{z \in \mathbb{Z}_4^\beta : \exists x, y \in \mathbb{Z}_4^\beta, \epsilon_1 x + \epsilon_2 y + \epsilon_3 z \in C_\beta\}, \end{aligned} \tag{3}$$

where $\epsilon_1 = -1 + u + u^2, \epsilon_2 = 1 - u^2$ and $\epsilon_3 = 1 - u$. It is clear that $C_{\beta,1}, C_{\beta,2}$ and $C_{\beta,3}$ are linear codes of length β over $\mathbb{Z}_4$ and C_β can be expressed as

$$C_\beta = (-1 + u + u^2)C_{\beta,1} + (1 - u^2)C_{\beta,2} + (1 - u)C_{\beta,3}$$

Theorem 3: Let C_β be a linear code of length β over R. Then $\Phi(C_\beta) = C_{\beta,1} \otimes C_{\beta,2} \otimes C_{\beta,3}$, and $|C_\beta| = |C_{\beta,1}||C_{\beta,2}||C_{\beta,3}|$.

Proof: Let $C_\beta = (-1+u+u^2)C_{\beta,1} + (1-u^2)C_{\beta,2} + (1-u)C_{\beta,3}$ where $C_{\beta,1}, C_{\beta,2}, C_{\beta,3}$ are as defined in (3). Let $d \in \Phi(C_\beta)$. Then there exists $(-1+u+u^2)a + (1-u^2)b + (1-u)c \in C_\beta$ such that $d = \Phi((-1+u+u^2)a + (1-u^2)b + (1-u)c) = \Phi((-a+b+c) + (a-c)u + (a-b)u^2) = (a, b, c)$. It gives $d \in C_{\beta,1} \otimes C_{\beta,2} \otimes C_{\beta,3}$, and so $\Phi(C_\beta) \subseteq C_{\beta,1} \otimes C_{\beta,2} \otimes C_{\beta,3}$. Conversely, let $(a, b, c) \in C_{\beta,1} \otimes C_{\beta,2} \otimes C_{\beta,3}$. Since $a \in C_{\beta,1}, b \in C_{\beta,2}$ and $c \in C_{\beta,3}$, we have $d = (-1 + u + u^2)a + (1 - u^2)b + (1 - u)c \in C_\beta$. Then $\Phi(d) = \Phi((-1 + u + u^2)a + (1 - u^2)b + (1 - u)c) = \Phi((-a + b + c) + (a - c)u + (a - b)u^2) = (a, b, c)$, gives $(a, b, c) = \Phi(d) \in \Phi(C_\beta)$. Hence $\Phi(C_\beta) = C_{\beta,1} \otimes C_{\beta,2} \otimes C_{\beta,3}$. Also $|C_\beta| = |\Phi(C_\beta)|$, so $|C_\beta| = |C_{\beta,1}||C_{\beta,2}||C_{\beta,3}|$. ■

Theorem 4: Let $C_\beta = (-1+u+u^2)C_{\beta,1} \oplus (1-u^2)C_{\beta,2} \oplus (1-u)C_{\beta,3}$ be a linear code of length β over R. Then, C_β is a cyclic code if and only if $C_{\beta,j}$ is the cyclic code over $\mathbb{Z}_4$, for $j = 1, 2, 3$.

Proof: Let $(c_0, c_1, \ldots, c_{\beta-1}) \in C_\beta$, where $c_i = (-1+u+u^2)s_i + (1-u^2)y_i + (1-u)z_i$, for $i = 0, 1, \ldots, \beta-1$. Consider, $s = (s_0, s_1, \ldots, s_{\beta-1}), y = (y_0, y_1, \ldots, y_{\beta-1})$ and $z = (z_0, z_1, \ldots, z_{\beta-1})$. Then $s \in C_{\beta,1}, y \in C_{\beta,2}, z \in C_{\beta,3}$. Suppose that C_β is a cyclic code, then we have $(c_{\beta-1}, c_0, \ldots, c_{\beta-2}) \in C_\beta$. Thus, $(c_{\beta-1}, c_0, \ldots, c_{\beta-2}) = (-1 + u + u^2)(s_{\beta-1}, s_0, \ldots, s_{\beta-2}) + (1 - u^2)(y_{\beta-1}, y_0, \ldots, y_{\beta-2}) + (1-u)(z_{\beta-1}, z_0, \ldots, z_{\beta-2})$. Therefore, $(s_{\beta-1}, s_0, \ldots, s_{\beta-2}) \in C_{\beta,1}, (y_{\beta-1}, y_0, \ldots, y_{\beta-2}) \in C_{\beta,2}$ and $(z_{\beta-1}, z_0, \ldots, z_{\beta-2}) \in C_{\beta,3}$, which implies that $C_{\beta,j}$ is the cyclic code over $\mathbb{Z}_4$, for $j = 1, 2, 3$. On the other hand, suppose that linear codes $C_{\beta,1}, C_{\beta,2}$ and $C_{\beta,3}$ are cyclic codes of length β over $\mathbb{Z}_4$. Then, $(s_{\beta-1}, s_0, \ldots, s_{\beta-2}) \in C_{\beta,1}, (y_{\beta-1}, y_0, \ldots, y_{\beta-2}) \in C_{\beta,2}$ and $(z_{\beta-1}, z_0, \ldots, z_{\beta-2}) \in C_{\beta,3}$. Now, $c_i = (-1 + u + u^2)(s_{\beta-1}, s_0, \ldots, s_{\beta-2}) + (1 - u^2)(y_{\beta-1}, y_0, \ldots, y_{\beta-2}) + (1 - u)(z_{\beta-1}, z_0, \ldots, z_{\beta-2}) \in$

$(-1+u+u^2)C_{\beta,1} \oplus (1-u^2)C_{\beta,2} \oplus (1-u)C_{\beta,3} = C_\beta$, which implies that C_β is a cyclic code of length β over R. ■

As similar to Propositions 18 and 19 of [4] we have the following proposition.

Proposition 1: Let $C_\beta = (-1+u+u^2)C_{\beta,1} \oplus (1-u^2)C_{\beta,2} \oplus (1-u)C_{\beta,3}$ be a cyclic code of length β over R. Then there exists a polynomial $g(x) \in R[x]$ with $g(x)|(x^\beta - 1)$ such that $C = \langle g(x) \rangle$, where

$$g(x) = (-1+u+u^2)g_1(x) + (1-u^2)g_2(x) + (1-u)g_3(x)$$

and $g_1(x), g_2(x)$ and $g_3(x)$ are the generator polynomials of cyclic code $C_{\beta,1}$, cyclic code $C_{\beta,2}$ and cyclic code $C_{\beta,3}$, respectively.

Proof: Consider $C_{\beta,1}, C_{\beta,2}$ and $C_{\beta,3}$ are cyclic codes of length β over $\mathbb{Z}_4$ respectively, then we can assume that the generator polynomials of $C_{\beta,1}, C_{\beta,2}$ and $C_{\beta,3}$ are $g_1(x), g_2(x)$ and $g_3(x)$, respectively. Therefore,

$$(-1+u+u^2)g_1(x) \in (-1+u+u^2)C_{\beta,1} \subseteq C_\beta,$$

$$(1-u^2)g_2(x) \in (1-u^2)C_{\beta,2} \subseteq C_\beta$$

and

$$(1-u)g_3(x) \in (1-u)C_{\beta,3} \subseteq C_\beta,$$

Thus, $(-1+u+u^2)g_1(x)+(1-u^2)g_2(x)+(1-u)g_3(x) \in C_\beta$. On the other hand, let $f(x) \in C_\beta$. Since, $C_\beta = (-1+u+u^2)C_{\beta,1} \oplus (1-u^2)C_{\beta,2} + (1-u)C_{\beta,3}$, then there exists $s(x)g_1(x) \in C_{\beta,1}, v(x)g_2(x) \in C_{\beta,2}$ and $t(x)g_3(x) \in C_{\beta,3}$ such that $f(x) = (-1+u+u^2)s(x)g_1(x)+(1-u^2)v(x)g_2(x)+(1-u)t(x)g_3(x)$, where $s(x), v(x), t(x) \in \mathbb{Z}_4[x]$. Therefore, $f(x) \in \langle (-1+u+u^2)g_1(x)+(1-u^2)g_2(x)+(1-u)g_3(x) \rangle$. Thus, $C_\beta \subseteq \langle (-1+u+u^2)g_1(x) + (1-u^2)g_2(x) + (1-u)g_3(x) \rangle$, which implies that $C = \langle (-1+u+u^2)g_1(x) + (1-u^2)g_2(x) + (1-u)g_3(x) \rangle$. According to the theory of cyclic codes over finite field, we know that $g_1(x)|(x^\beta - 1), g_2(x)|(x^\beta - 1)$ and $g_3(x)|(x^\beta - 1)$. Therefore, for $j = 1, 2, 3$, there exist polynomials $h_j(x) \in \mathbb{Z}_4[x]$ such that

$$x^\beta - 1 = h_1(x)g_1(x)$$
$$x^\beta - 1 = h_2(x)g_2(x)$$
$$x^\beta - 1 = h_3(x)g_3(x),$$

which implies that $x^\beta - 1 = (-1+u+u^2)h_1(x)g_1(x) + (1-u^2)h_2(x)g_2(x) + (1-u)h_3(x)g_3(x) = [(-1+u+u^2)h_1(x) + (1-u^2)h_2(x) + (1-u)h_3(x)]g(x)$ Therefore, $g(x)$ is a divisor of $x^\beta - 1$. ■

According to the above Theorem, it is easy to get the following corollary and omit the proof process here.

Corollary 1: Let $C_\beta = (-1+u+u^2)C_{\beta,1} \oplus (1-u^2)C_{\beta,2} \oplus (1-u)C_{\beta,3}$ be a cyclic code of length β over R and $g_1(x), g_2(x)$ and $g_3(x)$ be the generator polynomials of $C_{\beta,1}, C_{\beta,2}$ and $C_{\beta,3}$, respectively. Then

$$|C_\beta| = 4^{3\beta - (\deg(g_1(x)) + \deg(g_2(x)) + \deg(g_3(x)))}$$

Definition 4: An $R-$submodule C of $\mathbb{Z}_4^\alpha R^\beta$ is called a cyclic code of length (α, β) if and only if $(e_{\alpha-1}, e_0, \ldots, e_{\alpha-2} \mid r_{\beta-1}, r_0, \ldots, r_{\beta-2}) \in C$ whenever $(e_0, \ldots, e_{\alpha-1} \mid r_0, \ldots, r_{\beta-1}) \in C$.

Theorem 5: Let C be a linear code over $\mathbb{Z}_4R$ of length (α, β), and let $C = C_\alpha \times C_\beta$, where C_α is linear code over $\mathbb{Z}_4$ of length α and C_β is linear code over R of length β. Then C is a cyclic code if and only if C_α is a cyclic code over $\mathbb{Z}_4$ and C_β is a cyclic code over R, respectively.

Proof: Let C be a $\mathbb{Z}_4R-$cyclic code of length (α, β) and $(e_0, \ldots, e_{\alpha-1} \mid r_0, \ldots, r_{\beta-1}) \in C$, where $(e_0, \ldots, e_{\alpha-1}) \in C_\alpha$ and $(r_0, \ldots, r_{\beta-1}) \in C_\beta$. As C is a $\mathbb{Z}_4R-$cyclic code, we get $(e_{\alpha-1}, e_0, \ldots, e_{\alpha-2} \mid r_{\beta-1}, r_0, \ldots, r_{\beta-2}) \in C$, which implies $(e_{\alpha-1}, e_0, \ldots, e_{\alpha-2}) \in C_\alpha$ and $(r_{\beta-1}, r_0, \ldots, r_{\beta-2}) \in C_\beta$. Therefore, C_α and C_β are cyclic codes of length α and β over $\mathbb{Z}_4$ and R, respectively.

Conversely, suppose that C_α and C_β are cyclic codes over $\mathbb{Z}_4$ and R, respectively. Let $(e_0, e_1, \ldots, e_{\alpha-1}) \in C_\alpha$ and $(r_0, r_1, \ldots, r_{\beta-1}) \in C_\beta$. Therefore, $(e_{\alpha-1}, e_0, \ldots, e_{\alpha-2}, r_{\beta-1}, r_0, \ldots, r_{\beta-2}) \in C_\alpha \times C_\beta = C$. Hence, C is a $\mathbb{Z}_4R-$cyclic code of length (α, β). ■

IV. Reversible cyclic codes over $\mathbb{Z}_4R$

In this section, we mainly study some properties of reversible codes.

Theorem 6: Let $C_\beta = (-1+u+u^2)C_{\beta,1} \oplus (1-u^2)C_{\beta,2} \oplus (1-u)C_{\beta,3}$ be a cyclic code of arbitrary length β over R. Then C_β is reversible code if and only if $C_{\beta,1}, C_{\beta,2}$ and $C_{\beta,3}$ are reversible codes, respectively, where $C_{\beta,1}, C_{\beta,2}$ and $C_{\beta,3}$ are cyclic codes over $\mathbb{Z}_4$.

Proof:

Let $C_{\beta,1}, C_{\beta,2}$ and $C_{\beta,3}$ be reversible which means $C_{\beta,1}^r, C_{\beta,2}^r, C_{\beta,3}^r \in C_\beta$. Then for any $b \in C_\beta$ we have $b = (-1+u+u^2)b_1 + (1-u^2)b_2 + (1-u)b_3$, where $b_1 \in C_{\beta,1}, b_2 \in C_{\beta,2}$ and $b_3 \in C_{\beta,3}$. We can easy know that $b_1^r \in C_{\beta,1}, b_2^r \in C_{\beta,2}$ and $b_3^r \in C_{\beta,3}$, thus $b^r = (-1+u+u^2)b_1^r + (1-u^2)b_2^r + (1-u)b_3^r \in C_\beta$. Hence C_β is reversible.

Conversely, if C_β is reversible code, then for any $b = (-1+u+u^2)b_1+(1-u^2)b_2+(1-u)b_3 \in C_\beta$, where $b_1 \in C_{\beta,1}, b_2 \in C_{\beta,2}$ and $b_3 \in C_{\beta,3}$, we have $b^r = (-1+u+u^2)b_1^r + (1-u^2)b_2^r + (1-u)b_3^r \in C_\beta$ and $b^r = (-1+u+u^2)b_1^r + (1-u^2)b_2^r + (1-u)b_3^r = (-1+u+u^2)c_1 + (1-u^2)c_2 + (1-u)c_3$, where $c_1 \in C_{\beta,1}, c_2 \in C_{\beta,2}$ and $c_3 \in C_{\beta,3}$. Then $(-1+u+u^2)(b_1^r - c_1) + (1-u^2)(b_2^r - c_2) + (1-u)(b_3^r - c_3) = 0$, thus $b_1^r = c_1 \in C_{\beta,1}, b_2^r = c_2 \in C_{\beta,2}$ and $b_3^r = c_3 \in C_{\beta,3}$. Hence $C_{\beta,1}, C_{\beta,2}$ and $C_{\beta,3}$ are reversible.

■

Theorem 7: Let C be a linear code over $\mathbb{Z}_4R$ of length (α, β), and let $C = C_\alpha \times C_\beta$, where C_α is linear code over $\mathbb{Z}_4$ of length α and C_β is linear code over R of length β. Then C is reversible if and only if C_α and C_β are reversible over $\mathbb{Z}_4$ and R, respectively.

Proof: Suppose $C = C_\alpha \times C_\beta$ is reversible and $s = (e_0, e_1, \ldots, e_{\alpha-1} \mid r_0, r_1, \ldots, r_{\beta-1}) \in C$ where $(e_0, e_1, \ldots, e_{\alpha-1}) \in C_\alpha$ and $(r_0, r_1, \ldots, r_{\alpha-1}) \in C_\beta$. Since C is reversible, then we have $s^r = (e_{\alpha-1}, e_{\alpha-2}, \ldots, e_1, e_0) \mid (r_{\beta-1}, r_{\beta-2}, \ldots, r_1, r_0) \in C$ which implies $(e_{\alpha-1}, e_{\alpha-2}, \ldots, e_1, e_0) \in C_\alpha$ and $(r_{\beta-1}, r_{\beta-2}, \ldots, r_1, r_1) \in C_\beta$. Thus C_α and C_β are

reversible over $\mathbb{Z}_4$ and R, respectively. Conversely, let $s = (e_0, e_1, \ldots, e_{\alpha-1}) \mid (r_0, r_1, \ldots, r_{\beta-1}) \in C$ where $(e_0, e_1, \ldots, e_{\alpha-1}) \in C_\alpha$ and $(r_0, r_1, \ldots, r_{\beta-1}) \in C_\beta$. Suppose that C_α and C_β are reversible over $\mathbb{Z}_4$ and R, respectively. Then $(e_{\alpha-1}, e_{\alpha-2}, \ldots$
$, e_1, e_0) \in C_\alpha$ and $(r_{\beta-1}, r_{\beta-2}, \ldots, r_1, r_0) \in C_\beta$. Thus $s^r = (e_{\alpha-1}, e_{\alpha-2}, \ldots, e_1, e_0) \mid (r_{\beta-1}, r_{\beta-2}, \ldots, r_1, r_0) \in C$. Therefore, C is reversible. ■

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