OPTIMAL CONTROL APPLIED TO ECONOMIC STABILIZATION

By Dr. Kada Akacem
Professeur à l'Institut des sciences économiques de l'université d'Alger

Optimal Control Applied to Economic Stabilization

By Dr. Kada Akacem
Professeur à l'Institut des sciences économiques de l'université d'Alger

ABSTRACT

The prevailing optimal control approach to economic stabilization theory deals with what is called fixed-time free endpoint problems. The purpose of this paper, and in order to avoid the drawbacks of the above approach, is to propose a different use of optimal control in economic stabilization by solving an inverted problem : a fixed endpoint, free-time problem.

Introduction

Since the article of Sengupta (1970), the optimal control approach to economic stabilization theory has invariably dealt with what is called fixed-time, free endpoint probems. in this type of optimazation problems, the control time, T, is fixed a priori and the final state of the economy is assumed to be free.

The disadvantage of this approach is that at the exogenously fixed final time, T, the state of the economy might very well be far from the equilibrium state, in either direction. In which case we will have an under shooting or an over shooting problem. The purpose of this paper is to propose a different use of optimal control in stabilization theory, by solving an inverted problem, that can be called a fixed endpoint, free-time probem. Here the final time, T, is assumed to be free, to be determined endogenously by the optimization process. instead, the final state of the economy is given a priori as equal to a given desired level... The concern here is to make sure that, by the end of the stabilization process the economic system reaches a given, a priori, target. In addition, to the old approach, here the optimization will be subject to, a control constraint, among other things. In the example to be studied, the goal is to find the time-path, and the total amount of, government expenditure necessary to transfer the economic system from a non-desired initial state to a desired final state.

I - The model used

For an illustration purpose only, we shall use the following dynamic disequilibrium, multiplier accelerator model :

$Z\ (t) = C\ (t) + I\ (t) + (t)$ (1)

$\ddot{y}\ (t) = \frac{1}{F}\ \{ y\,(t) \quad Z\ (t) \}$ (2)

Where the consumption C (t) is given by

$C\ (t) = (1 - s)\ y\ (t)$, s being the propensity to save,

and the investment I (t) is defined by :

$I\ (t) = vy\ (t)$, where v is the acceleration coefficient,

G (t) is an " official " demand, used as a control variable, to stabilize the economy, according to the dynamic adjustment mechanism between aggregrate demand and aggregate supply, as given by equation (2),

where F is the time constant of the production lag, and $y\ (t) = \frac{dy\ (t)}{dt}$

Z (t) is, the aggregate demand measured from the initial equilibrium value, and y (t) is the aggregate production (identified with national incomo) measured from the initial equilibrium level.

From above we get :

$Z = (1 - s)\ y + v\ddot{y} + G$ (3)

the variable t has been dropped for convenience, i.e., for example y stands for y (t).

Now some transformation on the variable G is needed in order to be able to apply optimal control theory to the probem under study. G will be the only control variable and it will be assumed that $\underline{G} \leq G \leq \underline{G}$

Where G is a lower level and G an upper level for government expenditure. Both G and G are constant.

Define g(t) by :

$$g\ (t) = 2\ \frac{G - \frac{1}{2}\ (G + G)}{G - \underline{G}} \qquad (4)$$

Then $\quad / g(t) / \leq 1 \quad$ (5)

We then get from above :

$$\ddot{y} = - ay + bg + c \quad (6)$$

Where

$$a = \frac{S}{F \quad V} \quad (7)$$

$$b = \frac{\overline{G} - G}{F \quad V} \quad (8)$$

$$c = \frac{\overline{G} + G}{F \quad V} \quad (9)$$

A change of origin $y \Rightarrow y' = y - \frac{c}{a}$ is now made so that equation (6) becomes

$$\dot{y}' = - ay' + bg \quad (10)$$

In the text of the paper we use y to mean this y'.

Equation (6)is a fundamental equation as it represents the dynamic of the economic system. As it is well known its solution is given by (see ref. 1) :

$$y(t) = y_0 e^{-at} + e^{-at} \int^{t}_{t} e^{-ai_0} [\, bg(t) \,] \, di \quad (10 \text{ bis})$$

Where i t is a (dummy) variable of integration.

II - The problem

a/ Find the control g (t) that would transfer a nonzero initial state y (0) = y_0 at time t = 0, to a zero final state y (T) = 0 at time t = T, such that the following performance functional :

$$I_2 = \frac{I}{2} \int^{t} \overset{2}{g}(t) \, dt \quad (11)$$

is minimized, subject to :

$$\overset{\circ}{y} = - ay + bg \quad (12)$$

with

$y(0) = y_0$

$y(T) = 0$

T is free

and the control constraint,

$[g] \leq 1$ (13)

(we assume that y (0) = y is transferable to y (T) = 0)

First let us form the hamiltonian function :

$$H = \frac{1}{2} g^2 + p(-ay + bg) \quad (14)$$

b - Necessary conditions

b. 1 - The minimum principle gives :

$$H\{y^*, g^*, p^*, t\} \leq H\{y^*, g, p^*, t\} \quad (15)$$

$$\frac{1}{2}g^{*2} + p^*bg^* - p^*ay^* \leq \frac{1}{2} g^2 + p^*bg - p^*ay^* \quad (16)$$

OR

$$\frac{1}{2}g^{*2} + p^*bg^* \leq \frac{1}{2}g^2 + p^*bg \quad (17)$$

b. 2 - The canonical system is given by :

$$H_p^* = \dot{y}^* = -ay^* + bg^* \quad (18)$$

$$-H_y^* = p^* = ap^* \quad (19)$$

with $y(0) = y_0$

and $y(T) = 0$

b. 3 - Since the hamiltonian function, as given by equation (14) does not depend explicitly on time and the final time T is free, we have, again, the following additional necessary condition.

The hamiltonian function is equal to zero on the optimal trajectory, during all the time-interval (0, T), that is :

$$H \{ y^*, g^*, p^* \} = 0 \qquad (20)$$

Then

$$\frac{1}{2} g^{*2} + p^* \{ - ay^* + bg^* \} = 0 \qquad (21)$$

c - The possible optimal policies

From equation (17) and for [g (t)] < 1, we will have an interior solution for the minimum of the hamiltonian function. In this case the boundaries of the control constraint.

$[g] \leq 1$

do not affect the solution. The necessary and sufficient conditions to minimize the hamiltonian function (14), are then :

$$\frac{\partial H}{\partial g^*} = g^* + p^*b = 0 \qquad (22)$$

$$\frac{\partial^2 H}{\partial g^{*2}} = 1 > 0 \qquad (23)$$

Equation (23) assures us that the optimal control g^* (t) will **minimize** the hamiltonian function (14).

Condition (22) gives us the optimal control as :

$$g^* = -p^*b \qquad (24)$$

when

$[g] < 1$

which amounts to :

$$g^* = - p^*b \qquad \text{if } [p^*b] < 1 \qquad (25)$$

What happens then when

$$[p^*b] \geq 1 \tag{26}$$

In this case nad taking into account the control constraint

$[g] \leq 1$ which implies $[g^*] \leq 1$

the optimal stabilization policy is :

$$g^* = \begin{cases} 1 & \text{if} \quad p^*b \leq -1 \\ -1 & \text{if} \quad p^*b \geq 1 \end{cases} \tag{27}$$

in order for the hamiltonian to be minimum.

Combining equations (25) and (27), the minimum principle shows that the optimal government expenditure $g^*(t)$ is such that :

$$g^* = \begin{cases} -1 & \text{if} \quad p^*b \geq -1 \\ -p^*b & \text{if} \quad [p^*b] \leq 1 \\ 1 & \text{if} \quad p^*b \leq -1 \end{cases} \qquad \text{for all } t \tag{28}$$

or

$$g^* = -\,\text{sat}\,(p^*b) \tag{29}$$

where we define the saturation (sat) function as :

$$\text{sat } f = \begin{cases} f & [f] < 1 \\ \text{sgn} f & [f] \geq 1 \end{cases} \tag{30}$$

From condition (29) we have :

$$p^* = p\,(0)e^{at} \tag{31}$$

Which gives us the possible situations as given below : (see also fig. 1, 2, 3, 4).

$$g^* = 1 \qquad \text{if} \qquad p(0) \leq -1 \qquad \text{for all } \; t \tag{32}$$

$$g^* = \left[\begin{array}{l} -p^*b \text{ for } -1 < p\,(0) < 0 \text{ and } T < t_1 \text{ and } t \in \{0, t_1\} \\ \qquad \text{OR} \\ \left[\begin{array}{l} 1 \text{ fa } t \in \{t_1, T\} \\ \qquad \text{for } -1 < p\,(0) < 0 \text{ and } T \geq t_1 \\ \text{and } -p^*b \text{ for } t \ni \{0, t_1\} \end{array}\right. \end{array}\right. \tag{33}$$

$$g^* = \left[\begin{array}{l} -p^*b \text{ for } 0 < p\,(0) < 1 \text{ and } T < t_2 \text{ and } t \in \{0, t_2\} \\ \qquad \text{OR} \\ \left[\begin{array}{l} -p^*b \text{ for } t \in \{0, t_2\} \\ \qquad \text{for } 0 < p\,(0) < 1 \text{ and } T > t_2 \\ \text{and } -1 \text{ for } t \in \{t_2, T\} \end{array}\right. \end{array}\right. \tag{34}$$

$$g^* = -1 \quad \text{for} \quad p\,(0) \geq 1 \quad \text{for all } \; t \,\&\, \{0, T\} \tag{35}$$

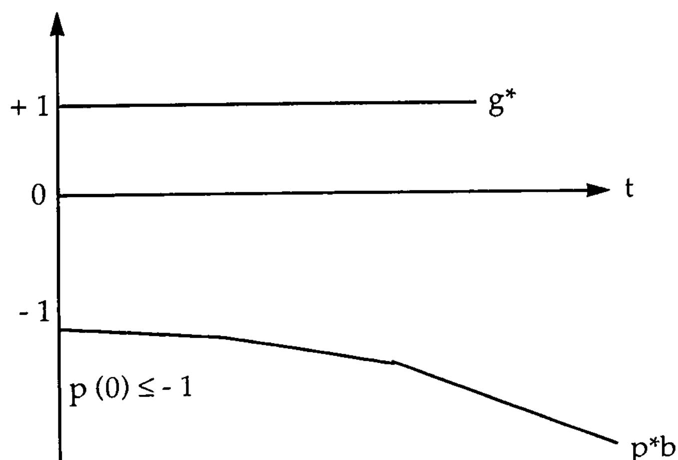

Fig. 1

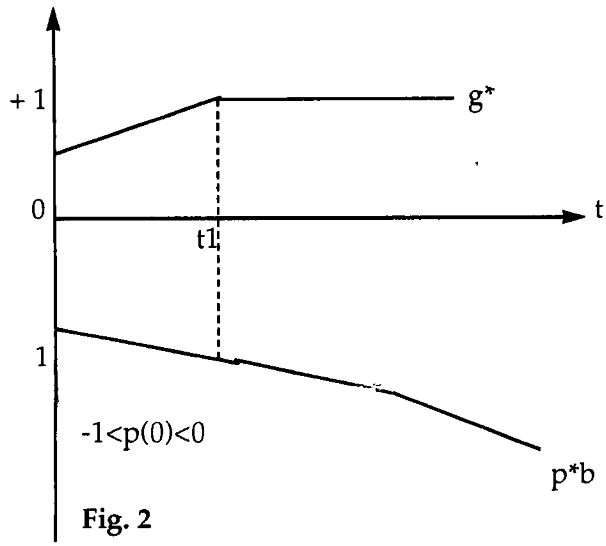

Fig. 2

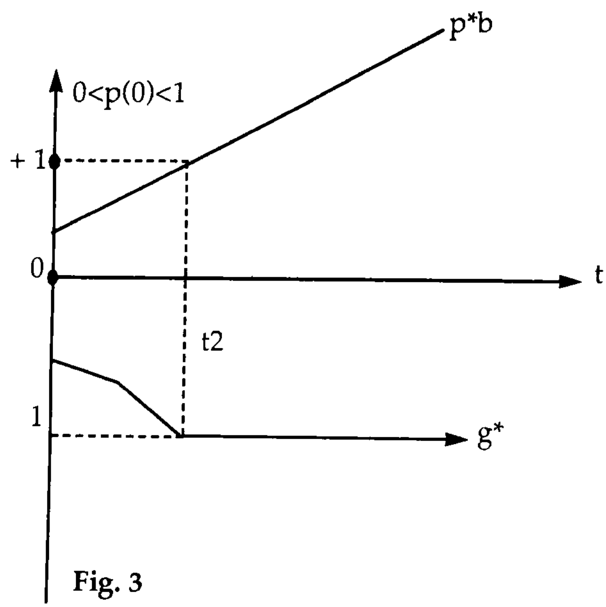

Fig. 3

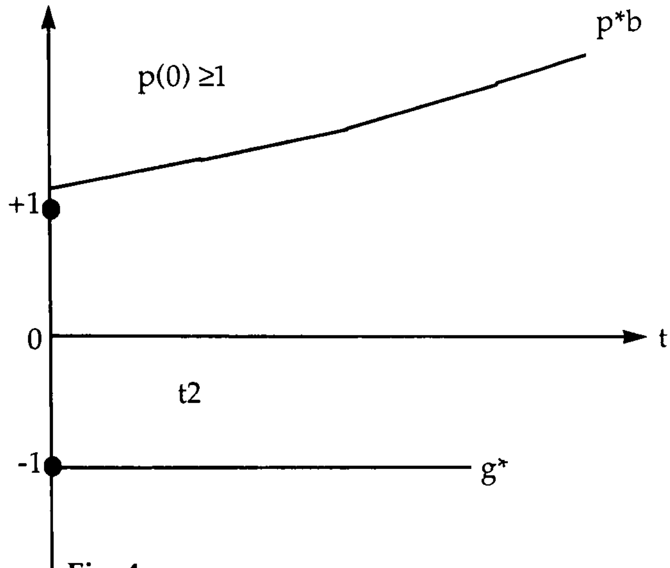

Fig. 4

Where :

for the set (33) , t_1 is such that :

$$p^* (t_1)\, b = 1 \tag{36}$$

$$p^* (t)\, b < - 1 \qquad \text{for } t > t_1$$

and for the set (34), t_2 is such that

$$p^* (t_2)\, b = 1 \tag{37}$$

$$P^* (t)\, b > 1 \qquad \text{for } t > t_2$$

the case $g^* = 0$ gives $y^* (t) = y_0 e^{-at}$ so that $y^* (T) = 0$

requires $T = \infty$, which we exclude.

Notice that statements (33) and (34) would eventually give policies in two phases that would result in some control expenditure saving. Indeed in these cases the control variable magnitude reaches its highest possible values (- 1 or 1) only after a certain time, and for a part only, of the stabilization time, unlike the bang-bang stabilization policy where the policy variable is equal to one (or the other) of its constraint boundaries during all the stabilization horizon. We already get here some feeling about the importance of the timing of a policy in economic stabilization. Indeed statements (33) and (34) would tell us **when** and for **how long** which policy should be used.

d. The Optimal Policy

In this paragraph we will try to see which policy, among the possible ones, is optimal for each possible situation of the initial state y (0) (positive or neagative). In other words we shall see which policy (or policies) would transfer positive initial states to the origin and which would transfer negative initial states of the economy. We shall do this in the following way :

d. 1 - Case : $g^* = 1$

When $g^* = 1$

$$y^*(t) = y_o e^{-at} + b(1 - e^{-at}) \qquad (38)$$

Which Vanishes only when $y < 0$ since $b > 0$.

But $y_o < 0$ implies y (t) negative which tells us that the control $g^*(t) = 1$ can do the job only when $y(t) < 0$.

d. 2 - Case : $g^* = -1$

Similarly, it can be shown that the policy $g^* = -1$ can transfer only positive (initial) states of the economy to the origin.

d. 3 - Case : $g^* = -p^*b$, $T < t_1$, or $t_1 \in (0, T)$

As can easily be shown, this policy can do the job only for y (t) negative. When y (t) is positive, the policy $g^* = -p^*b$, $T < t_2$ can be used.

d. 4 - Case :

$$g^* \begin{cases} -p^*b \text{ for } t \in (0_1, t_2) \text{ and } -1 < p(0) < 0, \ T \geq t_1 \\ 1 \text{ for } t \in [t_1, T] \end{cases} \qquad (39)$$

for $t \in [t_1, T]$ that is for $t \geq t$, we have

$$y^*(t)\ y_0 e^{-at} + b e^{-at} \left(\int_o^{t_1} K e^{2at} dt + \int_{t_1}^{t} e^{at} dt \right), \quad t > t^1 \qquad (40)$$

Where $K = -bp(0) > 0$.

When $y^*(T) = 0$ is imposed, one ends up with

$$y^*(t) = -\frac{b}{a}\left(e^{a(T-t)} - 1\right)$$

Which shows that y (t) is negative since $\frac{b}{a}$ is positive and,

$e^{a(T-t)} > 1$ for $t > 0$ and $t \in \{ t_1, T\}$

Remember that equation (41) is valid for $t \geq t_1$ so that we write :

$$y(t) = -b/a\ (e^{a(T-t)} - 1) \quad \text{for } t = t1 \tag{42}$$

and

$$y(t) > -b/a\ (e^{a(T-t)} - 1) \quad \text{for } t > t_1 \tag{43}$$

since y (t) < 0 and a (T - t) is decreasing as $t \longrightarrow T$

From all the above results we deduce :

$$g^* = \begin{cases} 1 & \text{if } -b/a\ (e^{a(T-t_1)} - 1) \leq y(t) < 0 \qquad t \geq t_1 \\ -p^*b & \text{if } y(t) < -b/a\ (e^{a(T-t_1)} - 1) < 0 \qquad t < t_1 \end{cases} \tag{44}$$

d. 5 - Case :

$$g^* = \begin{cases} -p^*b \text{ for } t\ \varepsilon \{ 0, t_2 \} & \\ \text{and} & \text{when } 0 < p(0) < 1,\ T \geq t_2 \\ -1 \quad \text{for } t \in \{ t_2, T\} & \end{cases}$$

We then have :

$$g^* = \begin{cases} -p^*b \text{ if } -b/a\ (e^{a(T-t_2)} - 1) < y(t), t < t_2 \\ -1 \quad \text{if } 0 < y(t) \leq -b/a\ (e^{a(T-t_2)} - 1), t \geq t_2 \end{cases} \tag{46}$$

Combining all cases together we get the following optimal control strategy :

$$g^* = \begin{cases} -p^*b & \text{for } y(t) > -\ b/a\ \{ e^{a(T-t_2)} - 1\} > 0 \\ -1 & \text{for } 0 < y(t) < -b/a\ \{ e^{a(T-t_2)} - 1\} \\ -p^*b & \text{for } y(t) < -\ b/a\ \{ e^{a(T-t_1)} - 1 \} < 0 \\ +1 & \text{for } -\ b/a\ \{ e^{a(T-t_1)} - 1 \} \leq y < 0 \end{cases} \tag{47}$$

Conclusion

The above results, and more specifically expressions (44) and (46), show the strong dependence of the optimal policy on the choice of the final time.

Depending on the final time, the **form** (not only the value) of the optimal policy can be totally different.

In this later case the solution is given in a closed-loop (feedback) form as $g^*(t)$ is now a function of the state variable $y^*(t)$, which can be very useful for economic stabilization purposes.

The optimal strategy (47) tells us, also, about the trade off between the stabilization time and the stabilization control. For a minimum time stabilization policy, we need to apply a control (a bang-bang control) of the " strongest " possible value (here either $g^*= 1$ or $g^*= +1$) throughout the stabilization horizon, while if we are willing to allow more time to the economy to return to its initial equilibrium we need apply, as strategy (47) show, the control $g^*(t) = 1$, for instance, during part of the stabilization time only (from t, to T) and the control $g^*(t) = -p^*b < 1$ for the rest of the time. The value of the optimal time T as well as switching times t_1 and t_2, can be derived, by making use of equation (22) and of the boundary condition $y(T) = 0$. Another difference with the prevailing approach, is that the assumption of a magnitude constraint on the control variable gives the possibility of corner, just as well as, interior solutions. This control constraint, that does exist in the real world, has also an effect on the controllability(1) of the system ; this is the reason for the assumption made after equation (13).

---

(1) a future paper will spell out the details of this effects.

References

1 - Athans M. and Falb, P.L. (1966), Optimal Control,

An Introduction to the theory and its application, New York : Mc Graw-Hill Book. Co.

2 Pontryagin L.S., et al., (1962) : The Mathematical Théory of Optimal Processes, New York, Wiley.

3 - Sengupta J.K., (1970), "Optimal Stabilization Policy with a Quadratic Criterion Function " Review of Economic Studies, pp. 127-145.

4 Turnovsky, S.T., (1973), "Optimal Stabilization Policies for Deterministic and Stochastic LinearSystems ", Review of Economic Studies, XL, p. 79-96.