LAMB SHIFT IN HYDROGEN-LIKE ATOM INDUCED FROM NON-COMMUTATIVE QUANTUM SPACE-TIME

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Abstract

In this work we present an important contribution to the non-commutative approach to the hydrogen atom to deal with lamb shift corrections. This can be done by studying the Klein-Gordon equation in a non-commutative space-time as applied to the Hydrogen atom to extract the energy levels, by considering the second-order corrections in the non commutativity parameter and by comparing with the result of the current experimental results on the Lamb shift of the 2P level to extract a bound on the parameter of non-commutativity. Phenomenologically we show that the non-commutativity effects induce lamb shift corrections.

Keywords: *Klein-Gordon equation, non-commutative space, quantum Hydrogen atom.*

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I. INTRODUCTION

In recent years many arguments have been suggested to motivate a deviation from the flat-space concept at very short distances [1, 2] so that we have a new concept of quantum spaces. Quantum spaces depend on parameters such that for a particular value of these parameters they become the usual flat space. We consider the simplest version of quantum spaces as the natural extension of the usual quantum mechanical commutation relations between position and momentum, by imposing further commutation relations between position coordinates themselves. The non-commutative space can be obtained by the coordinate operators where we replace the ordinary product by star product: $\left[x^{\mu}, x^{\nu}\right] = i\theta^{\mu\nu}$

Our motivation is to study the effect of non-commutativity at the level of quantum mechanics when time-space non-commutativity is accounted for. One can study the physical consequences of this theory by making detailed analytical estimates for measurable physical quantities and compare the results with experimental data to find an upper bound on the θ parameter. In this work we present an important contribution to the non-commutative approach to the hydrogen atom. Our goal is to solve the Klein-Gordon equation for the Coulomb potential in a non-commutative space-time up to second-order in the non-commutativity parameter using the Seiberg-Witten maps and the Moyal product. We thus find the non-commutative modification of the energy levels of the hydrogen atom and we show that the non-commutativity is the source of lamb shift corrections.

The purpose of this paper is to study the extension of the Klein-Gordon field in canonical non- commutative time-space by applying the result obtained to a hydrogen atom. This paper is organized as follows. In section 2, we propose an invariant action of the non-commutative boson field in the presence of an electromagnetic field. In section 3, using the generalised Euler-Lagrange field equations, we derive the deformed Klein-Gordon (KG) equation for the hydrogen atom. We solve these deformed equation and obtain the noncommutative modification of the energy levels. Furthermore, we derive the nonrelativistic limit of the non-commutative KG equation for a hydrogen atom and solve it using perturbation theory. Finally, in section 4, we draw our conclusions.

II. ACTION

We consider an action for a free boson field in the presence of electrodynamic gauge field in a non-commutative space-time. We propose the following action [28].

$$S = \int d^4x \left(L_{MB} - \frac{1}{4} F_{\mu\nu} * F^{\mu\nu} \right) \tag{1}$$

Where L_{MB} is the boson matter densitie in the non-commutative space-time and is given by:

$$L_{MB} = \eta^{\mu\nu} (\hat{D}_{\mu}\hat{\varphi})^{+} * (\hat{D}_{\nu}\hat{\varphi}) + m^2 \hat{\varphi}^{+} * \hat{\varphi} \tag{2}$$

where the gauge covariant derivative is defined as: $\hat{\mathrm{D}}_{\mu} = \partial_{\mu} + ie\hat{A}_{\mu}$.

From the action variational principle the generalised equations of Lagrange up to second order of θ are [29]:

$$\frac{\partial L}{\partial \hat{\varphi}} - \partial_{\mu} \frac{\partial L}{\partial \left(\partial_{\mu}\hat{\varphi}\right)} + \partial_{\mu}\partial_{\nu} \frac{\partial L}{\partial \left(\partial_{\mu}\partial_{\nu}\hat{\varphi}\right)} + \mathrm{O}\left(\theta^2\right) = 0 \tag{3}$$

Where:

$$L = L_{MB} - \frac{1}{4} F_{\mu\nu} * F^{\mu\nu} \tag{4}$$

III. NON-COMMUTATIVE TIME-SPACE KG EQUATION

Using the modified field equation (3) , with the generic boson field $\hat{\varphi}$ one can find in a free non-commutative space-time and in the presence of the external potential $\hat{A}_\mu$ the following modified Klein-Gordon equation:

$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu - m_e{}^2\right)\hat{\varphi} + \left(ie\eta^{\mu\nu}\partial_\mu\hat{A}_\nu - e^2\eta^{\mu\nu}\hat{A}_\mu * \hat{A}_\nu + 2ie\eta^{\mu\nu}\hat{A}_\mu\partial_\nu\right)\hat{\varphi} = 0 \tag{5}$$

Where the deformed external potential $\hat{A}_\mu\begin{pmatrix} -e/r \\ 0 \end{pmatrix}$ in free non-commutative space-time is [30]:

$$\hat{a}_0 = -\frac{e}{r} - \frac{e^3}{r^4}\theta^{0k}x_k + \mathrm{O}(\theta^2). \qquad \hat{a}_i = \frac{e^3}{4r^4}\theta^{ik}x_k + \mathrm{O}(\theta^2) \tag{6}$$

for a non-commutative time-space, $\theta^{0k} \neq 0$ and $\theta^{ik} = 0$, where i; k = 1; 2; 3. In this case, we can check that

$$\eta^{\mu\nu}\partial_\mu\partial_\nu = -\partial_0{}^2 + \Delta \tag{7}$$

And

$$2ie\eta^{\mu\nu}\hat{A}_\mu\partial_\nu = i\frac{2e^2}{r}\partial_0 + 2i\frac{e^4}{r^4}\theta^{0j}x_j\partial_0 \tag{8}$$

$$-e^2\eta^{\mu\nu}\hat{A}_\mu * \hat{A}_\nu = \frac{e^4}{r^2} + 2\frac{e^5}{r^5}\theta^{0j}x_j$$

then the Klein-Gordon equation (20) up to $\mathrm{O}(\theta^2)$ takes the form:

$$\left(-\partial_0{}^2 + \Delta - m_e{}^2\right)\hat{\varphi} + \left(\frac{e^4}{r^2}{}_\nu + i\frac{2e^2}{r}\partial_0 + 2i\frac{e^4}{r^4}\theta^{0j}x_j\partial_0 + 2\frac{e^5}{r^5}\theta^{0j}x_j\right)\hat{\varphi} = 0 \tag{9}$$

The solution to eq. (9) in spherical polar coordinates (r,ϑ,ϕ) takes the separable form:

$$\hat{\varphi}(r,\vartheta,\phi,t) = \frac{1}{r}\,\hat{\mathrm{R}}(\mathrm{r})\hat{Y}(\vartheta,\phi)\exp(-iEt) \tag{10}$$

Then eq. (9) reduces to the radial equation:

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)-e^4}{r^2} + \frac{2Ee^2}{r}\right)\hat{R} + \left(E^2 - m_e^2 + 2E\frac{e^4}{r^4}\theta^{0j}x_j + 2\frac{e^5}{r^5}\theta^{0j}x_j\right)\hat{R} = 0 \tag{11}$$

In eq.(11) the coulomb potential in non-commutative space-time appears within the perturbation terms [31]:

$$H^\theta_{pert} = 2E\frac{e^4}{r^4}\theta^{0j}x_j + 2\frac{e^5}{r^5}\theta^{0j}x_j, \tag{12}$$

where the first term is the electric dipole–dipole interaction created by the non-commutativity, the second term is the electric dipole–quadruple interaction. These interactions show us that the effect of space-time non-commutativity on the interaction of the electron and the proton is equivalent to an extension of two nuclei interactions at a considerable distance. This idea effectively confirms the presence of gravity at this level. To investigate the modification of the energy levels by eq. (12), we use the first-order perturbation theory. The spectrum of H_O and the corresponding wave functions are well-known and given by:

$$R_{nl}(r) = \sqrt{\frac{a}{n+\nu+1}}\left(\frac{n!}{\Gamma(n+2\nu+2)}\right)^{1/2} x^{\nu+1} .e^{-x/2}L_n^{2\nu+1}(x). \tag{13}$$

where the relativistic energy levels are given by:

$$E_{nl}{}^0 = \frac{m_e\left(n_{+1/2} + \sqrt{(l_{+1/2})^2 - \alpha^2}\right)}{\sqrt{(n_{+1/2})^2 + (l_{+1/2})^2 + 2n_{+1/2}\sqrt{(l_{+1/2})^2 - \alpha^2}}} \tag{14}$$

and $L_n^{2\nu+1}$ are the associated Laguerre polynomials [32], with the following notations:

$$\nu = -\frac{1}{2} + \sqrt{(l_{+1/2})^2 - \alpha^2},\ \alpha = e^2, \quad a = \sqrt{m_e^2 - E_{nl}^{02}} \quad n_{+1/2} = n + 1/2,\ l_{+1/2} = l + 1/2 \tag{15}$$

III. 1 NON-COMMUTATIVE CORRECTIONS OF THE RELATIVISTIC ENERGY

Now to obtain the modification to the energy levels as a result of the terms (12) due to the non-commutativity of space-time we use perturbation theory. For simplicity, first of all, we choose the coordinate system (t,r,ϑ,ϕ) so that $\theta^{0i} = -\theta^{i0} = \theta\delta^{01}$; such that $\theta^{0i}x_i = \theta r$ and assume that the other components are all zero and also the fact that in first-order perturbation theory the expectation value of r^{-3} and r^{-4} are as follows:

$$\left\langle nlm \middle| r^{-3} \middle| nlm' \right\rangle = \int_0^\infty R_{nl}^2 r^{-1} dr \delta_{mm'}$$
$$= \frac{2a^3}{\nu(2\nu+1)(n+\nu+1)}\left[1+\frac{n}{\nu+1}\right]\delta_{mm'} \tag{16}$$
$$= f(3)$$

$$\left\langle nlm \middle| r^{-4} \middle| nlm' \right\rangle = \int_0^\infty R_{nl}^2 r^{-2} dr \delta_{mm'}$$
$$= \frac{4a^4}{(2\nu-1)\nu(2\nu+1)(n+\nu+1)}\left\{1+\frac{3n}{\nu+1} \right. \tag{17}$$
$$\left. + \frac{3n(n-1)}{(\nu+1)(2\nu+3)}\right\}\delta_{mm'} = f(4),$$

Now, the correction to the energy to first order in Theta is:

$$E^{\theta(1)} = \left\langle H_{pert}^{\theta} \right\rangle_{nlm} \tag{18}$$

wher H_{pert}^{θ} is the non-commutative corrections perturbation Hamiltonian, which is given in the following relation:

$$H_{pert}^{\theta} = 2E\frac{e^4}{r^3}\theta + 2\frac{e^5}{r^4}\theta \tag{19}$$

To calculate $E^{\theta(1)}$, we use the results in equations (16) and (17), to obtain:

$$E^{\theta(1)} = 2\theta\alpha^2\left[E_{nl}^0 f(3) + \alpha f(4)\right] \tag{20}$$

Finally the energy correction of the hydrogen atom in the framework of the non-commutative KG equation is:

$$\Delta E^{NC} = \frac{E^{\theta(1)}}{2E_{nl}^0} = \theta\alpha^2\left[f(3) + \frac{\alpha}{E_{nl}^0} f(4)\right] \tag{21}$$

This result is important because it reflects the existence of Lamb shift, which is induced by the non-commutativity of the space. Obviously, when $\theta = 0$; then $\Delta E^{NC} = 0$, which is exactly the result of the space-space commuting case, where

the energy-levels are not shifted. We showed that the energy-level shift for 1S is:

$$\Delta E_{1S}^{NC} = \theta\alpha^2\left[f_{1S}(3) + \frac{\alpha}{E_{10}^0} f_{1S}(4)\right] \tag{22}$$

In out analysis, we simply identify spin up if the non-commutativity parameter takes the eigenvalue $+\theta$ and spin down if non-commutativity parameter takes the eigenvalue $-\theta$. Also we can say that the Lamb shift is actually induced by the space-time non-commutativity which plays the role of a magnetic field and spin in the same moment (Zemann effect). This represents Lamb shift corrections for $l = 0$. This result is very important: as a possible means of introducing electron spin we replace $l \to \pm(j + 1/2)$ and $n \to n - j - 1 - 1/2$ where j is the quantum number associated to the total angular momentum.

Then the $l = 0$ state has the same total quantum number $j = 1/2$. In this case the non-commutative value of the energy levels indicates the splitting of $1S$ states.

III.2 NON-RELATIVISTIC LIMIT

The non-relativistic limit of the non-commutative K-G equation (11) is written as [33, 34]:

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1) - e^4}{r^2} + \frac{2m_e e^2}{r} + 2m_e\varepsilon + 2m_e\frac{e^4}{r^3}\theta + 2\frac{e^5}{r^4}\theta\right)\hat{R} = 0 \tag{23}$$

In this non-relativistic limit the charged boson does not represent a single charged particle, but is a distribution of positive and negative charges which are different and extended in space linearly in $\sqrt{\theta}$. The absence of a perturbation term of form θ / r^2 in the non-commutative coulomb interaction demonstrates that the distribution of positive and negative charges is spherically symmetric.
This can be interpreted as the spherically symmetric distribution of charges of the quarks inside in the proton.
Now to obtain the modification of energy levels as a result of the non-commutative terms in eq. (23), we use the first-order perturbation theory. The spectrum of $H_0(\theta = 0)$ and the corresponding wave functions are well-known and given by:

$$\varepsilon_n = \frac{m_e\alpha^2}{2\hbar^2 n^2} \tag{24}$$

And

$$R_{nl}(r) = \frac{1}{n}\left(\frac{(n-l-1)!}{a(n+l)!}\right)^{1/2} x^{l+1} e^{-x/2} L_{n-l-1}^{2l+1}(x). \tag{25}$$

where $x = \frac{2r}{an}$ and $a = \hbar^2 / (m_e\alpha)$ is the Bohr radius of the Hydrogen atom. The coulomb potential in non - commutative space-time appears within the perturbation terms:

$$H_{pert}^{\theta} = 2\theta\alpha^2\left(\frac{m_e}{r^3} + \frac{\alpha}{r^4}\right) \tag{26}$$

where the expectation value of r^{-3} and r^{-4} are as follows:

$$\left\langle nlm \middle| r^{-3} \middle| nlm' \right\rangle_{l\succ 0} = \frac{2}{a^3 n^3 l(l+1)(2l+1)}\delta_{mm'} \tag{27}$$
$$= g(3)$$

$$\left\langle nlm\left|r^{-4}\right|nlm'\right\rangle_{l\succ 0} = \left\{\frac{4\left(3n^3 - l(l+1)\right)}{a^4 n^5 (2l-1) l (l+1)(2l+1)(2l+3)} + \frac{35\left(3n^3 - l(l+1)\right)}{3(l-1)(2l-1)(l+2)(2l+1)(2l+3)}\right\}\delta_{mm'} = g(4) \tag{28}$$

Hence the modification to the energy levels is given by:

$$\Delta E^{NC} = \theta\alpha^2\left[g_{nl}(3) + \frac{\alpha}{m_e} g_{nl}(4)\right] + \mathrm{O}\left(\theta^2\right) \tag{29}$$

We can also compute the correction to the Lamb shift of the 2P level where we have:

$$\Delta E^{NC}(2P) = 0.243156\theta(MeV)^3 \tag{30}$$

According to ref. [35] the current theoretical result for the lamb shift is 0:08 kHz. From the splitting (30), this then gives the following bound on θ:

$$\theta \leq (8.5TeV)^{-2} \tag{31}$$

This corresponds to a lower bound for the energy scale of $8.5TeV$, which is in the range that has been obtained in refs [36, 37, 38, 39], namely $1-10TeV$.

IV. CONCLUSION

Using the Seiberg-Witten maps and the Moyal product up to second order in the non-commutativity parameterθ, we have derived the deformed KG equation for a hydrogen atom in non-commutative space-time. By solving the deformed KG equation we found the energy shift up to the second order of θ, which proofs that the non-commutativity has an effect similar to that of the Lamb Shift. After that we have obtained the non-relativistic limit of the non-commutative KG equation for a Coulomb potential. We then compared the corrections induced to energy levels by this non-commutative effect to experimental results from high precision hydrogen spectroscopy, and obtained the bound for the parameter of non-commutativity around $(8.5TeV)^{-2}$.

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