AN ASSUMED STRAIN BASED ON TRIANGULAR ELEMENT WITH DRILLING ROTATION

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ABSTRACT

The lack of compatibility between degrees of freedom of various elements is a problem frequently encountered in practice during modeling complex structures. Coupling of membrane and beam elements is an illustrating classical example. The problem is generally treated through an additional rotational degree of freedom [1]. In this respect a new element based on the strain field has been developed with a drilling rotation in the Bergan sense [2]. The triangular element, with three nodes and three degrees of freedom constructed in this way, presents very good performance and may be used in various practical problems.

KEY WORDS: Triangular element, drilling rotation, strain model, membrane, plane elasticity

1 INTRODUCTION

The strain based approach was applied by Sabir and Ashwell [3] to develop, a new class of elements for general plane elasticity problems in Cartesian coordinates.

A basic rectangular element having the only essential nodal degrees of freedom and satisfying the requirements of strain free rigid body modes and compatibility within element is first developed by [4], this eight degrees of freedom element is based on linear direct strains and constant shear strain is named **SBRIE** (**S**train **B**ased **R**ectangular **I**nplane **E**lasticity). Other element meeting the above basic consideration together with equilibrium within the element is also developed **(SBRIEE).** Several variations of the **(SBRIE)** are also suggested by [5] and used under names (**SBRIE1),** and **(SBRIE2)** to produce solutions to general plane elasticity problems, these variations are based on satisfying equilibrium at the element level and the use of statical condensation. With the continuation of the development of the strain based approach many elements for general plane elasticity as well as shells have been devised by Sabir et al.

Membrane elements -usually require only two degrees of freedom at each node, this type of elements is simple to derive and has been widely used. However some membrane structures such as shear walls and plates with holes are usually combined with beams having three essential degrees of freedom, if in the combined structures these three degrees of freedom are used they would adequately represent the beam element and they will necessitate the introduction of an additional degree of freedom in the two dimensional plane elements. This additional degree of freedom can be presented by the inplane rotation or drilling rotation of the plane element.

The problem of the inclusion of the inplane rotation as an additional degree of freedom has also been treated by [6] using strain approach, and a simple and efficient rectangular element including the inplane rotation is derived **SBRIEIR** (**S**train **B**ased **R**ectangular **I**nplane **E**lasticity **I**nplane **R**otation**),** and a triangular element incorporating the inplane rotation **SBTIEIR** (**S**train **B**ased **T**riangular **I**nplane **E**lasticity **I**nplane **R**otation**),** is also developed, these elements are non conforming.

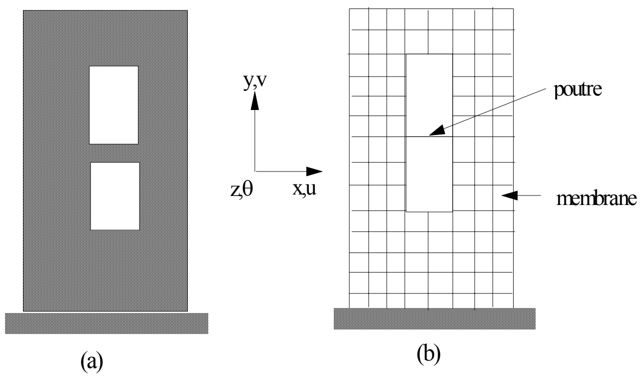

Figure 1: Structure (shear wall) and its discretization

The success of the application of the strain approach to two dimensions plane elasticity problems in Cartesian coordinates prompted the extension of the work to the development of strain based sector elements, Belarbi has developed a sector element with inplane rotation [7].

The object of this article is to improve the element of Sabir by proposing a new triangular element baptized **SBT3** (**S**train **B**ased **T**riangular **3**-node) based on the strain model with three degrees of freedom (dof) per node, including plane rotation as an additional (dof) (Drilling rotation).

2 DESCRIPTION OF ELEMENT ' SBTIEIR'

The strain displacement relationship inplane elasticity is given by:

$$\varepsilon X = U,x = (\partial U/\partial x)\ ;\ \varepsilon y = V,y\ ;\ \gamma xy = U,y + V,x \qquad (1)$$

We first integrate equation (1) with all the strains equal to zero, thus obtaining

$$\begin{aligned} U_R &= a_1 - a_3\, y \\ V_R &= a_2 + a_3\, x \\ \phi_z &= a_3 \end{aligned} \qquad (2)$$

The assumed strains are [6]:

$$\begin{aligned} \varepsilon_x &= a_4 + a_5 y + a_7 x \\ \varepsilon_y &= a_6 + a_7 x + a_5 y \\ \gamma_{xy} &= a_8 + a_9 (x + y) \end{aligned} \qquad (3)$$

After integrations of equations (3) we obtain;

$$\begin{aligned} U &= a_4 x + a_5 xy + a_7 \frac{(x^2 - y^2)}{2} + a_8 \frac{y}{2} + a_9 \frac{y^2}{2} \\ V &= a_5 \frac{(y^2 - x^2)}{2} + a_6 y + a_7 xy + a_8 \frac{x}{2} + a_9 \frac{x^2}{2} \\ \theta_z &= -a_5 x + a_7 y + a_9 \frac{(x - y)}{2} \end{aligned} \qquad (4)$$

The final displacement field will be obtained by combination of (4) and (2):

$$\begin{aligned} U &= a_1 - a_3 y + a_4 x + a_5 xy + a_7 \frac{(x^2 - y^2)}{2} + a_8 \frac{y}{2} + a_9 \frac{y^2}{2} \\ V &= a_2 + a_3 x + a_5 \frac{(y^2 - x^2)}{2} + a_6 y + a_7 xy + a_8 \frac{x}{2} + a_9 \frac{x^2}{2} \\ \theta_z &= a_3 - a_5 x + a_7 y + a_9 \frac{(x - y)}{2} \end{aligned} \qquad (5)$$

It was revealed that the unsatisfactory results obtained by using these elements could be due to the unnecessary coupling between the direct strains.

In this work, a modified form of triangular element incorporating the inplane rotation, by avoiding such linking between the strains is developed

3 VARIATIONAL FORMULATION OF ELEMENT 'SBT3'

Figure 2 shows the geometry of element **SBT3** and the corresponding nodal displacements. Each node (i) with Ui, VI, and inplane rotation ϕ_{zi}

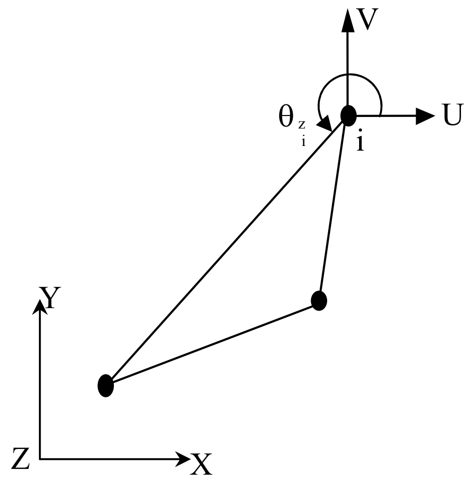

Figure 2 : SBT3 element with drilling rotation

Consider the triangular element shown in figure 2. The three Cartesian components of the field of strain in the plan of a point P of the element are given in equation (1).

U, V: components of displacement respectively in the directions x and y.

The components of the strain given in equations (1) can not be considered independent, for they are in terms of two displacements U, V and hence the strains must satisfy an additional equation called the compatibility equations. This equation can be formed by eliminating U, V from equation (1), hence:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0 \qquad (6)$$

By equating the strains, in equations (1) to zero and integrating the resulting partial differential equation, they obtained:

$$\begin{aligned} U &= a_1 - a_{3\,y} \\ V &= a_2 + a_{3\,x} \end{aligned} \qquad (7)$$

The inplane rotation θ_z is obtained by:

$$\theta_z = \frac{1}{2}\left(V,_x - U,_y\right) = a_3$$

This equation (7) gives the three components of rigid body displacements and requires three independent constants (a_1, a_2, and a_3).

The present element has three nodes and three dof by node (U, V, and ϕ_z). The field of displacement must contain nine independent constants. Three of them (a_1, a_2, and a_3) are already used to represent the MRB. The remaining six constants are available for expressing the deformation of the element. These are apportioned among the strains as below (8) so that the compatibility equation (6) is satisfied:

$$\begin{aligned} \varepsilon_x &= a_4 + a_5 y \\ \varepsilon_y &= a_6 + a_7 x \\ \gamma_{xy} &= a_8 + a_9 \left(x^2 + y^2\right) \end{aligned} \tag{8}$$

Expressions (8) are equated to the corresponding expressions in terms of U, V from equations (1) and the resulting equations integrated, to give

$$\begin{aligned} U_S &= a_4 x + a_5 xy - a_7 \frac{y^2}{2} + a_8 \frac{y}{2} + a_9 \frac{y^3}{3} \\ V_S &= -a_5 \frac{x^2}{2} + a_6 y + a_7 xy + a_8 \frac{x}{2} + a_9 \frac{x^3}{3} \\ \theta_z &= a_5 x - a_7 y + a_9 \frac{\left(y^2 - x^2\right)}{2} \end{aligned} \tag{9}$$

The complete shape function is the sum of the corresponding expressions from equations (7), and (9).

$$\begin{aligned} U &= a_1 - a_3 y + a_4 x + a_5 xy - a_7 \frac{y^2}{2} + a_8 \frac{y}{2} + a_9 \frac{y^3}{3} \\ V &= a_2 + a_3 x - a_5 \frac{x^2}{2} + a_6 y + a_7 xy + a_8 \frac{x}{2} + a_9 \frac{x^3}{3} \\ \theta_z &= a_3 + a_5 x - a_7 y + a_9 \frac{\left(y^2 - x^2\right)}{2} \end{aligned} \tag{10}$$

We notice that, the final functions of displacement contain quadratic and cubic terms thus allowing the change of curvature.

If the classic formulation is adopted, two problems can arise: geometrical problem of distortion for certain finite elements of higher degree (loss of precision) and problem of blocking or locking of membrane and shearing for the case of the finite elements of degree relatively low. The adoption of a model of strain [8] followed by a method of integration purely analytical would make it possible to avoid these two problems.

4 AUTOMATIC EVALUATION OF THE MATRIX [K$_0$]

The evaluation of the elementary matrix of rigidity is summarized with the evaluation of the following expression:

$$[\mathrm{Ke}] = \left[A^{-1}\right]^T \left[\iint_S [Q]^T [D].[Q].dx.dy\right] \left[A^{-1}\right] \tag{11a}$$

$$[K_e] = [A^{-1}]^T [K_0] [A^{-1}] \tag{11b}$$

With:

$$[K_0] = \iint_S [Q]^T [D].[Q].\mathrm{d}x.\mathrm{d}y \tag{11c}$$

Being given that [A] and its reverse can be evaluated numerically, it is carried out here as the evaluation of the integral (11c) becomes the key of the problem. Giving that, for certain elements, a too great distortion can lead to erroneous numerical results in particular in the calculation of Jacobien. We formulated for that an expression general, easy to implement on computer, allowing the evaluation in an automatic way the matrix [K$_0$] whatever the degree of the polynomial (10) and the distortion of the element (Figure 3).

In general, it is required to calculate double integrals of the form:

$$I = [K0] = \iint_S C.x^\alpha y^\beta \,\mathrm{d}x.\mathrm{d}y \tag{12}$$

The general form of the expression (12) is:

$$I = \sum_{P=1}^{2} I_P$$

With:

$$I_P = \frac{C}{\beta+1} \sum_{k=1}^{\beta+2} \frac{1}{k+\alpha} .C(k) \left(a_j^{\,k-1}.b_j^{\,\beta+2-k} - a_i^{\,k-1}.b_i^{\,\beta+2-k}\right) \left(x_n^{\,k+\alpha} - x_m^{\,k+\alpha}\right) \tag{13}$$

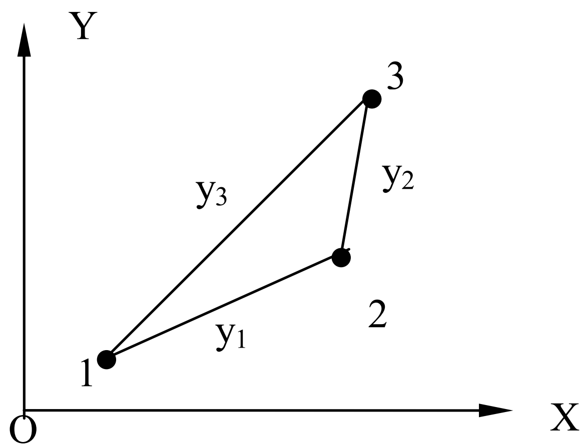

Figure 3: Distorted element (SBT3)

5 NUMERICAL RESULTS

Numerical results for a variety of problems of plane elasticity are presented to demonstrate the level of accuracy attainable with the present element **(SBT3).**

5.1 Higher order Patch test: Pure bending of a cantilever beam

It is useful to know how behaves a finite element displaying an Important geometrical distortion. Sze, Chen and Cheung studied this Problem [9] in order to test the performance and the robustness of elements 07β and 07β*.

Three examples are presented in this section in order to illustrate the interest of the model of strain. The element thus developed **(SBT3)**, provided with degrees of freedom of rotation, and is particularly robust and more powerful than the **SBTIEIR** and classical elements.

Consider a cantilever beam with rectangular section (L x t x h = 10 x 1 x 2) deformed in pure bending by two nodal forces (P=1000) forming a couple (consisting loading case).

The loading case (figures 4) represents the higher patch-test [10].

The cantilever beam is discretized by two rectangular elements of membrane (regular grid) or trapezoidal (distorted grid); various cases of boundary conditions [9] are shown in the figures 4a, 4b and 4c. The results obtained with elements **SBTIEIR** and **SBT3** are compared with the analytical solution given by [12].

Note:

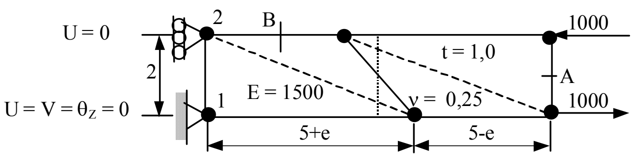

Figure 4a of a cantilever beam; Data and grids. Rotation θ_Z is free into 2.

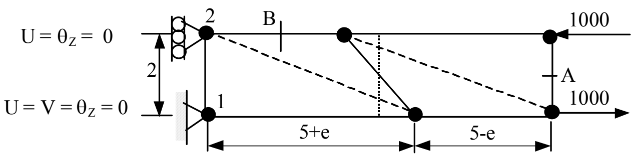

Figure 4b of a cantilever beam; Data and grids. Rotation θ_Z is fixed into 1 and 2.

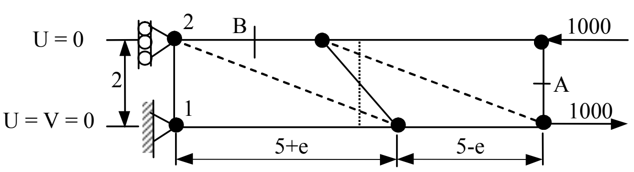

Figure 4c of a cantilever beam; Data and grids. Rotation θ_Z is free into 1, and 2.

Table 1a: Pure bending of a cantilever beam; regular and distorted grid: case of the figure 4a

e	**SBTIEIR [6]**		Present element **SBT3**	
	W_A	**σ_{xB}**	**W_A**	**σ_{xB}**
0	44.77	335	95.82	3016
0.5	44.51	230	96.37	3000
1	45.99	430	96.78	3000
1.5	46.60	479	97.04	3058
2	48.83	464	97.15	3058
2.5	48.39	419	97.13	3048
3	48.84	377	97.02	3035
3.5	49.01	340	96.87	3024
4	49.14	313	96.70	3014
4.5	49.04	293	96.51	3007
Ref. [12]	**100**	**3000**	**100**	**3000**

Table 1b: Pure bending of a cantilever beam; Regular and distorted grid: case of the figure 4b

E	**SBTIEIR** [6]		Present element **SBT3**	
	W_A	**σ_{xB}**	**W_A**	**σ_{xB}**
0	44.74	326	95.86	2976
0.5	44.98	215	96.45	2990
1	40.50	350	96.91	3015
1.5	46.51	454	97.14	3031
2	47.51	431	97.15	3058
2.5	48.15	380	97.12	3022
3	48.53	332	96.96	3010
3.5	48.68	292	96.76	2999
4	48.65	262	96.52	2990
4.5	48.48	240	96.29	2983
Ref. [12]	**100**	**3000**	**100**	**3000**

Table 1c: Pure bending of a cantilever beam; Regular and distorted grid: case of the figure 4c

E	**SBTIEIR [6]**		Present element **SBT3**	
	W_A	**σ_{xB}**	**W_A**	**σ_{xB}**
0	45.08	241	96.02	3018
0.5	45.33	230	96.60	3030
1	45.84	355	97.04	3051
1.5	46.88	479	97.30	3066
2	47.96	464	97.40	3066
2.5	48.68	419	97.38	3056
3	49.15	377	97.26	3044
3.5	49.40	340	97.10	3032
4	49.47	313	96.90	3023
4.5	49.40	293	96.70	3016
Ref. [12]	**100**	**3000**	**100**	**3000**

For the case of the regular grid (Table 1a; e=0), good results are obtained for element **SBT3** contrary to the element **SBTIEIR** which gives unacceptable results. On the other hand for the case of the distorted grid characterized by the distance "e" (e>0), the results of **SBT3** are powerful and comparable to the analytical solution. Element **SBTIEIR** remains sensitive to the distortions of the grid. The precision is always largely insufficient (Table 1a and 1b).

In the case of the figure 4b, the robustness of this element via the regular and distorted grid is confirmed. The Tables

1a and 1b show stability, the reliability and the good performance of this element **SBT3**, and whatever the geometrical distortion (only one element on h!). That is explained probably partly by the nature of analytical integration carried out. The distortion has a considerable influence on elements **SBTIEIR**. (Table1). These results confirm that the element thus developed satisfied good the higher patch ([10] and [11]).

Table 1c confirms the good performance and the stability of this element SBT3 contrary at element SBTIEIR.

5.2 Slender cantilever beam of MacNeal

Consider the slender cantilever beam of MacNeal and Harder [15] with rectangular section (6 x 2 x 1) deformed in pure bending by end moment (M=10) and by a load applied at the free edge (P=1).

The cantilever beam is modeled by six elements of membrane rectangular (figure 5a), trapezoidal (figure 5b) and parallelograms (figure 5c).

MacNeal [16] affirms that the trapezoidal shape of the membrane finite elements has four nodes without degrees of freedom of rotation (with linear fields) generate a locking even if these elements pass the patch test. It qualified this problem of "trapezoidal locking ".

NOTE: This rule does not apply for the strain based element. The results obtained for **SBT3** are compared with those obtained with other known quadrilateral elements (table2).

Through these three cases of grids (figures 5a, 5b, 5c), we underlined the effectiveness of this element **SBT3**. The results obtained for elements Q4 and PS5β (table 2) show well the problem of trapezoidal locking announced by MacNeal et al. [16].

In conclusion, we can say that element **SBT3** is very powerful for this type of problem dominated by bending. It remains stable with the geometrical distortions.

Table 2: Slender cantilever beam of MacNeal; Displacement standardized at the free end: case of figure 6

Element	Pure Flexure			Force shearing at the free end		
	Regular	Trapezoidal	Parallel	Regular	Trapezoidal	Parallel
Q4	0,093	0,022	0,031	0,093	0,027	0,034
PS5β	1,000	0,046	0,726	0,993	0,052	0,632
AQ	0,910	0,817	0,881	0,904	0,806	0,873
MAQ	0,910	0,886	0,890	0,904	0,872	0,884
Q4S [MAC 89]	-	-	-	0,993	0,986	0,988
07β	1,000	0,998	0,992	0,993	0,988	0,985
SBT3	0,989	0,988	0,988	0,964	0,950	0,950
SBTIEIR [6]	0.118	0.004	0.101	0.047	0.0005	0.036
Theory of the beams	**1,000 (0,270)**			**1,000 (0,1081)**		

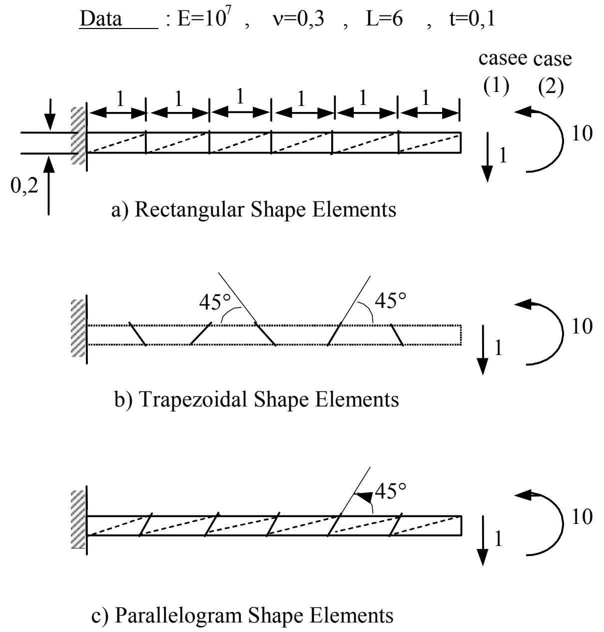

Figure 5: Slender cantilever beam of MacNeal. Data and grids.

5.3 Plane flexure of beam cantilever.

A beam cantilever, with uniform cross-section, carries a uniform vertical load (figure.6), calculate the deflection V_A at the free end.

This problem was dealt with by Batoz and Dhatt in their work [11] in order to test the performances of elements CST, LST, Q4, Q4WT 17, 18, Q4PS 19 and Q8. Récemment Ayad [20] made a similar study to test the reliability of these new elements FRQ and FRT based on the concept" Fibre Planes in Rotation". The results obtained for different grids are presented on table 3.

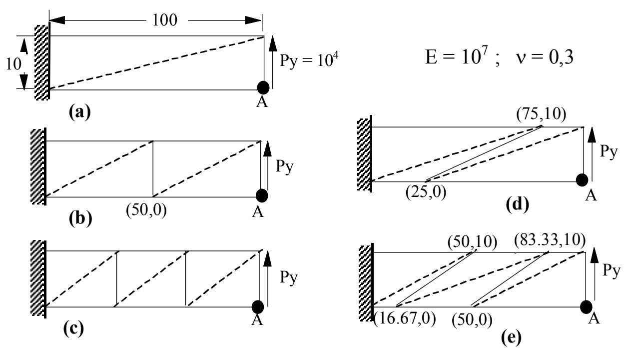

Figure 6: Grids regular and distorted

5.3.1 Comments: grids without distortions (Figures 6a, 6b and 6c)

The results obtained for **SBT3** are powerful and comparable with the robust element Q8.

Element **SBT3** converges better than CST, it is comparable with element LST in term of a total number of DOF

5.3.2 Comments: grids with distortions (Figures 6d and 6e)

Very good performance of element **SBT3** is confirmed. The corresponding results are more precise than elements FRT, CST, Q8 and are also comparable with element LST in term of the total number of DOF.

Table 3: Beam in plane flexure, Displacement V_A. Comparison with various elements

Figure	FRT	Q8	LST	CST	**SBT3**	**SBTIEIR** [SAB 85]
	IR: 1PH	IE :3x3	IE :3PH	IE :1PH	IA	IA
2.16a	2,32* (12)**	3,03 (16)	3,00 (18).	0,05 (8)	2,8846 (12)	1.42
2.16b	2,92 (18)	3,70 (26)	3,70 (30)	0,13 (12)	2,8993 (18)	1.68
2.16c	3,07 (24)	3,84 (36)	3,84 (42)	0,25 (16)	2,9289 (24)	1.69
2.16d	1,99 (18)	0,64 (26)	3,02 (30)	0,06 (12)	2,9155 (18)	1.42
2.16e	2,02 (24)	1,76 (36)	3,09 (42)	0,10 (16)	2,9660 (24)	1.40

*VA: Vertical displacement in A; IE: Exact integration; **NDLT: Number total dof;
IR: Reduced integration; IA: Integration analytical

The corresponding results are very close to those obtained with the regular grids. In conclusion, we can say that element **SBT3** is very powerful for this type of problem dominated by the flexure. The precision of element **SBTIEIR** is always largely insufficient.

6 CONCLUSION

While making it possible to combine various finite elements the ones with the others in the complex structures, the addition of degree of freedom of rotation Z also makes it possible to improve the precision without increasing the number of nodes and to remove the difficulties related to famous the sixth degree of freedom of the hulls.

Very good results are obtained. **SBT3** particularly robust (rich in membrane), is simplified much more and more powerful than element **SBTIEIR**. Currently the strain based element gain ground, would be this only because they make it possible to enrich the field by displacements by intermediate terms of a nature raised without introduction of nodes.

It interesting to explore the combination of **SBT3** with elements rich in flexure such as DKT, etc.

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